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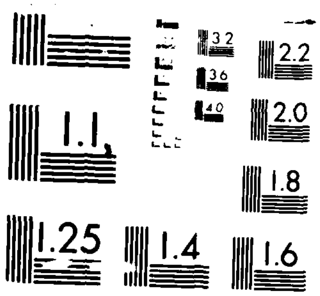

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ON THE EXISTENCE AND CONVERGENCE OF PROBABILITY

MEASURES ON CONTINUOUS SEMI-LATTICES

by

Tommy Norberg

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ON THE EXISTENCE AND CONVERGENCE OF PROBABILITY MEASURES ON CONTINUOUS SEMI-LATTICES

by

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1. Introduction

In this paper we investigate some aspects of probability measures on a large class of partially ordered sets, which satisfy a continuity property which extends the following well-known fact for the real line R:

$$x=\sup\{y\in R;\ y<x\},\qquad -\infty<x\leq\infty.$$

They are called continuous semi-lattices.

The extended real line $(-\infty,\infty]$ is a continuous semi-lattice and so is also the collection of all closed sets in a locally compact second countable Hausdorff space S. The collection of all compact sets in S is another example of such a partially ordered set, and if $S=R^d$ for some $d\in N=\{1,2,\ldots\}$, also the collection of all compact and convex sets is a continuous semi-lattice. Many sets of functions are continuous semi-lattices too. For instance the collection of all upper semi-continuous functions on S into $\bar{R}=[-\infty,\infty]$ and the collection of all capacities on S.

Thus the results of this investigation has rather a wide range of applicability. The investigation is primarily concerned with the question of existence of probability measures on continuous semi-lattices and continuous lattices, the latter being a special case of the former. The related questions of weak convergence and infinite divisibility of distributions are treated too.

The main result of this paper characterizes the collection of all probability measures on a fixed continuous semi-lattice. In the special case of a continuous lattice the

characterization is in terms of distribution functions. This existence result extends a theorem of Choquet [2] identifying the distributions of all random closed sets in a locally compact second countable Hausdorff space. Cf Matheron [11]. By applying it to the real line we obtain the well-known one-to-one correspondence between distributions of random variables and distribution functions, thereby explaining the similarity between this fundamental fact and Choquet's existence theorem.

As noted above the collection of all compact sets in a locally compact second countable Hausdorff space is a continuous semi-lattice and by applying the existence theorem to this particular partially ordered set we obtain two completely new sets of existence criteria for distributions of random compact sets. We furthermore obtain a new existence criterium for distributions of random compact convex sets in Euclidean spaces.

Of course the list of applications of the existence theorem can be made much longer. We leave this to the reader and to forthcoming publications dealing with special cases.

All the results on weak convergence of probability measures on continuous semi-lattices are w r t the Lawson topology and it should be noted that there may be other natural choices of topologies, especially when the semi-lattice under consideration has some further structure. This is a question that we plan to return to in a forthcoming publication.

Among other things our investigation showed that weak convergence can be characterized by pointwise convergence of the corresponding distribution functions on a sufficiently large

subset. This is a well-known result for random variables. Now we know that it holds also for many different kinds of random sets.

We have also defined and investigated a notion of infinite divisibility for distributions of random variables in continuous semi-lattices. Our results here generalize those known for random closed sets. See [11].

The titles of the subsequent sections are as follows:

2. Continuous partially ordered sets
3. Measurability
4. Existence and uniqueness
5. Convergence
6. Infinite divisibility
7. Applications to random set theory

2. Continuous partially ordered sets

In this section we review the notion of a continuous poset and discuss some examples of such objects. A general reference to continuous lattices - a slightly more narrow subject - is the monograph Giertz, Hofmann, Keimal, Lawson, Mislove & Scott [5]. We also review some relevant but not so widely known notions from topology.

Consider a non-empty set L endowed with a transitive, reflexive and anti-symmetric relation $\leq$. Such a set is called a poset, which is short for partially ordered set, and we refer to $\leq$ as the (partial) order on L. Note that any non-empty subset of L itself is a poset under the same order. Unless otherwise is stated directly, the order on the poset(s) under consideration will always be denoted $\leq$.

A mapping f between two posets is increasing (resp decreasing) if $x\leq y$ implies $f(x)\leq f(y)$ (resp $f(y)\leq f(x)$). A surjection f between two posets is an isomorphism if $x\leq y$ is equivalent to $f(x)\leq f(y)$. Two posets are isomorphic if they are connected by an isomorphism.

Note that on L there is an opposite relation $\leq^*$, called the reverse order, defined by

$$x\leq^* y \quad \text{iff} \quad y\leq x.$$

Of course also $\leq^*$ orders L, and we write L^* for the set L endowed with the reverse order $\leq^*$. The isomorphic posets L^{**} and L are always identified.

Let $A\subseteq L$. An upper bound of A is a member $x\in L$ satisfying $y\leq x$ for all $y\in A$. If there exists an upper bound z

of A satisfying $z \leq x$ for every upper bound x of A, then it is referred to as <u>the least upper bound</u> or <u>the join of</u> A, since there is at most one, and denoted $\vee A$. We often write $\vee_\alpha x_\alpha = \vee\{x_\alpha\}$. <u>Lower bounds</u> and <u>greatest lower bounds</u> or <u>meets</u> are defined analogously. We write $\wedge A$ for the meet of A provided it exists. We further write $x_n \uparrow x$ if $x_1 \leq x_2 \leq \ldots \leq x = \vee_n x_n$, and $x_n \downarrow x$ if $x_n \uparrow x$ in L^*.

Note that, by the definition, $\vee\emptyset = \wedge L$ if L has a least member, a <u>bottom</u>, and $\wedge\emptyset = \vee L$ if L has a greatest member, a <u>top</u>.

A poset is <u>directed</u> (resp <u>filtered</u>) if every finite non-empty subset has an upper (resp lower) bound. A non-empty $F \subseteq L$ is a <u>filter</u> on L if it is filtered and if $\uparrow x \subseteq F$ whenever $x \in F$. Here and subsequently, $\uparrow x = \{y;\ x \leq y\}$. We also write $\downarrow x = \{y;\ y \leq x\}$. By a <u>chain</u> we understand a poset in which $x \leq y$ or $y \leq x$ for every pair (x,y) of members.

A <u>semi-lattice</u> is a poset in which every finite non-empty subset has a meet. In a <u>lattice</u> every such subset has both a join and a meet. A poset is <u>up-complete</u> if it is closed under directed joins (i e if every directed subset has a join) and <u>complete</u> if it is closed under arbitrary joins. Note that a complete poset is closed under arbitrary meets too. Thus a poset is complete iff it is a <u>complete lattice</u>.

Fix two members x,y of an up-complete poset L. We say that x is <u>way below</u> y, and write $x \ll y$, if, whenever $y \leq \vee D$ for a directed $D \subseteq L$, we have $x \leq z$ for some $z \in D$. Note that, if L is complete, then $x \ll y$ iff $y \leq \vee A$, $A \subseteq L$ imply $x \leq \vee B$ for some

finite $B \subseteq A$.

Recall [10] that a poset L is said to be *continuous* if it is up-complete and if $\{y;\ y \ll x\}$ is directed with join x for all $x \in L$. It should be clear to the reader what we mean by a *continuous semi-lattice*. However note that a *continuous lattice* always is assumed to be complete.

Suppose L is a continuous poset. A subset $U \subseteq L$ is *Scott open* if $\uparrow x \subseteq U$ whenever $x \in U$ and if $\vee D \in U$, $D \subseteq L$ directed imply $D \cap U \neq \emptyset$. The collection of all Scott open subsets of L is a topology. It is called the *Scott topology* and denoted Scott(L). A function on or into or between continuous posets is called *Scott continuous* if it is continuous w r t all the Scott topologies involved.

Let $A \subseteq L$. The reader may wish to verify that A is *Scott closed* (i e $A^c \in \mathrm{Scott}(L)$) iff $\downarrow x \subseteq A$ whenever $x \in A$ and A is closed under directed meets w r t L.

A filter on L is called *open* if it is Scott open. We write $\mathcal{L}$ or OFilt(L) for the collection of all open filters on L provided with the inclusion order $F_1 \leq F_2$ iff $F_1 \subseteq F_2$. It is not hard to see that $\cup \mathcal{D} \in \mathcal{L}$ if $\mathcal{D} \subseteq \mathcal{L}$ is directed. Hence $\mathcal{L}$ is up-complete and $\vee \mathcal{D} = \cup \mathcal{D}$ when $\mathcal{D}$ is as above.

Let $x, y \in L$ and $F, G \in \mathcal{L}$. Then

(2.1) $\qquad x \ll y$ iff $y \in H \subseteq \uparrow x$ for some $H \in \mathcal{L}$;

(2.2) $\qquad F \ll G$ iff $F \subseteq \uparrow z \subseteq G$ for some $z \in L$.

Moreover,

(2.3) $\qquad x \in F$ implies $x \in H \subseteq \uparrow z \subseteq F$ for some $(z, H) \in L \times \mathcal{L}$.

For a proof of (2.1)-(2.3) consult Lawson [10]. Here it is also

proved that $\mathcal{L}$ is a continuous poset and that the mapping

$$x \to \mathcal{F}_x = \{F \in \mathcal{L};\ x \in F\} \tag{2.4}$$

is an isomorphism between L and OFilt($\mathcal{L}$). This fact is called the *Lawson duality*, and $\mathcal{L}$ is sometimes in the literature referred to as the *Lawson dual* of L. Lawson (loc cit) also proves that $\mathcal{L}$ is a semi-lattice with a top iff L is so.

The equivalences (2.1)-(2.3) are very important. Below they are used often and without explicit reference.

Now suppose L is a semi-lattice, not necessarily continuous. A member $p \in L$ is called *prime* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Clearly the top of L is prime if it exists. The *spectrum* of L is the set of all non-top primes. It is denoted Spec(L). By the *hull* of a point $x \in L$ we understand the set

$$h(x) = \{p \in \mathrm{Spec}(L);\ x \leq p\}.$$

From the definition of primes we get

$$h(x \wedge y) = h(x) \cup h(y), \qquad x, y \in L.$$

Moreover, if the join of $A \subseteq L$ exists, then

$$h(\vee A) = \cap h(A).$$

We say that L is *order-generated by primes* if

$$x = \wedge h(x), \qquad x \in L.$$

If L is complete then the collection $\mathrm{Spec}(L) \backslash h(L)$ is a topology on Spec(L) (provided $\mathrm{Spec}(L) \neq \emptyset$ of course). It is called the *hull-kernel topology*.

Next consider an arbitrary topological space S. Write $\mathcal{G}$ for its topology, which is a poset under inclusion $\subseteq$. It is not hard to verify that $S \backslash \{s\}^- \in \mathrm{Spec}(\mathcal{G})$ for all $s \in S$. It follows that $\mathcal{G}$ is order-generated by primes. Note that,

whenever $G \in \mathcal{G}$,

$$G=\{s \in S;\ S\backslash\{s\}^- \in \mathrm{Spec}(\mathcal{G})\backslash h(G)\}.$$

Hence the mapping

$$s \to S\backslash\{s\}^-;\quad S \to \mathrm{Spec}(\mathcal{G}) \tag{2.5}$$

is continuous if the latter space is endowed with its hull-kernal topology. Cf [7].

Recall that S is called a TO <u>space</u> if $\{s\}^-=\{t\}^-$ implies $s=t$, i e if the mapping in (2.5) is injective. If this mapping is bijective, then S is said to be <u>sober</u>. Cf [8].

It is well-known that all Hausdorff spaces are sober: Suppose $G \in \mathrm{Spec}(\mathcal{G})$, where $\mathcal{G}$ is a Hausdorff topology on S. Then there is some $s \in S\backslash G$. If $t \neq s$, choose disjoint neighborhoods $G_1, G_2 \in \mathcal{G}$ such that $s \in G_1$ while $t \in G_2$. Then $G_1 \cap G_2 \subseteq G$. Hence $G_1 \subseteq G$ or $G_2 \subseteq G$. Since the former is ruled out by assumption we must have $t \in G$. Hence $G=S\backslash\{s\}$.

Recall [8] that the <u>saturation</u> of $A \subseteq S$ is the set

$$A^s=\cap\{G \in \mathcal{G};\ A \subseteq G\}.$$

Moreover, A is called <u>saturated</u> if $A=A^s$. It is easily seen that $K \subseteq S$ is compact iff K^s is so. Moreover, all subsets of a T1 space are saturated. (Since $s \in A^s$ iff $\{s\}^- \cap A \neq \emptyset$ [5].)

Let us agree to say that S is <u>locally compact</u> if $s \in G \in \mathcal{G}$ implies $s \in K^0 \subseteq K \subseteq G$ for some compact $K \subseteq S$. Clearly we may always assume here that K is saturated. It is not hard to see that if S is locally compact, then $\mathcal{G}$ is a continuous lattice in which $G_1 \ll G_2$ if $G_1 \subseteq K \subseteq G_2$ for some compact (and saturated) $K \subseteq S$. In this case we further have

$$\{G \in \mathcal{G};\ K \subseteq G\} \in \mathrm{OFilt}(\mathcal{G})$$

as soon as $K \subseteq S$ is compact (and saturated).

Now suppose that S is both locally compact and sober. Write $\mathcal{K}$ for the collection of all compact and saturated subsets of S in the exclusion order $\supseteq$. Hofmann and Mislove [8] proves that $\mathcal{K}$ is a continuous semi-lattice in which $K_1 \ll K_2$ iff $K_2 \subseteq K_1^{\circ}$. Note that the top $\emptyset$ of $\mathcal{K}$ is isolated in the sense $\emptyset \ll \emptyset$. Hence also $\mathcal{K} \setminus \{\emptyset\}$ is a continuous semi-lattice.

The paper [8] also proves that the mapping

(2.6) $$K \to \{G \in \mathcal{G};\ K \subseteq G\};\quad \mathcal{K} \to \mathrm{OFilt}(\mathcal{G})$$

is an isomorphism. By the Lawson duality, so is also the mapping

(2.7) $$G \to \{K \in \mathcal{K};\ K \subseteq G\};\quad \mathcal{G} \to \mathrm{OFilt}(\mathcal{K}).$$

Let us also note here that $\mathcal{K}$ is a continuous semi-lattice if S only is TO. Cf [8]. However, in the absence of sobriety the isomorphism above between $\mathcal{K}$ and $\mathrm{OFilt}(\mathcal{G})$ breaks down.

Any continuous poset L endowed with its Scott topology is a locally compact sober space [10]. To see the local compactness, suppose $x \in U \in \mathrm{Scott}(L)$. Choose $y \in U$ such that $y \ll x$. Then $x \in \{z;\ y \ll z\} \subseteq \uparrow y \subseteq U$. Now local compactness follows from the easily proved fact that $\uparrow y$ is compact w r t $\mathrm{Scott}(L)$. Note also that $(\uparrow y)^{\circ} = \{z;\ y \ll z\}$. A routine compactness argument next shows that $U_1 \ll U_2$ iff $U_1 \subseteq \bigcup_{i=1}^{n} \uparrow x_i \subseteq U_2$ for some finite sequence $x_1, \ldots, x_n \in L$.

Let us further say that S is <u>quasi</u> <u>locally</u> <u>compact</u> [8] [16] if $s \in G \in \mathcal{G}$ implies $s \in H \ll G$ for some $H \in \mathcal{G}$. Such spaces are called core-compact in [6], semi-locally bounded in [9] and spaces satisfying condition (C) in [3]. The monograph [5] discusses them too. It is not hard to see [6] that a space is

quasi locally compact iff its topology is continuous. Hence, if S is locally compact then S is quasi locally compact too. The converse is false, unless S is sober [7].

3. Measurability

Here we provide our continuous posets with a canonical σ-field. Then we discuss necessary and sufficient conditions for measurability.

Of course a successful discussion of probability measures on continuous posets require some condition of countability. Here it is convenient to assume the Scott topology to be second countable.

Let L be a continuous poset. We say that a subset $A \subseteq L$ is <u>separating</u> if $x \ll y$ implies the existence of some $z \in A$ satisfying $x \leq z \leq y$. It is not hard to see that $A \subseteq L$ is separating iff $\{y \in A;\ y \ll x\}$ is directed with join x for all $x \in L$.

PROPOSITION 3.1: *Let L be a continuous poset. The following five conditions are equivalent*:

(i) Scott(L) has a countable open base;

(ii) Scott($\mathcal{L}$) has a countable open base;

(iii) L contains a countable separating subset;

(iv) $\mathcal{L}$ contains a countable separating subset;

(v) there is a countable collection $\mathcal{B}$ of subsets of L such that, whenever $x \in F \in \mathcal{L}$, we have $\uparrow x \subseteq B \subseteq F$ for some $B \in \mathcal{B}$.

<u>Proof:</u> Suppose $\mathcal{B}$ fulfills the requirements of (v), which trivially follows from (i). Whenever $B, C \in \mathcal{B}$ is separated in the sense that $B \subseteq F \ll H \subseteq C$ for some $F, H \in \mathcal{L}$, choose $x_{BC} \in L$ such that

$B \subseteq \uparrow x_{BC} \subseteq C$. Clearly the obtained collection $A=\{x_{BC}\}$ is countable. Suppose $x \ll y$. Then $\uparrow y \subseteq F \ll H \subseteq \uparrow z \subseteq G \subseteq \uparrow x$ for some $F,H,G \in \mathcal{L}$ and some $z \in L$. Next choose $B,C \in \mathcal{B}$ with $\uparrow y \subseteq B \subseteq F$ and $\uparrow z \subseteq C \subseteq G$. Finally choose $x_{BC} \in A$ such that $B \subseteq \uparrow x_{BC} \subseteq C$. Then, obviously, $x \leq x_{BC} \leq y$. This shows (iii).

Similarly the reader may show that (iv) follows from (iii). That (iv) implies (i) is an immediate consequence of the fact that $\mathcal{L}$ is an open base for Scott(L).

This shows that (i), (iii), (iv) and (v) are equivalent. In fact it shows that (ii) and (iv) are equivalent too. QED

We write Σ or Σ_L for the σ-field on L generated by the sets $\uparrow x$, $x \in L$. The main result of this section gives several equivalent conditions for measurability. In particular it says that Σ coincides with the σ-field generated by Scott(L) if the latter is second countable.

<u>PROPOSITION 3.2: Let</u> $(\Omega,\mathcal{R})$ <u>be a measurable space and let</u> ξ <u>be a mapping of</u> Ω <u>into a continuous poset</u> L. <u>Suppose</u> Scott(L) <u>has a countable open base. Then the following four conditions are equivalent</u>:

(i) ξ is measurable $\mathcal{R}/\Sigma$;

(ii) $\{x \leq \xi\} \in \mathcal{R}$, $x \in L$;

(iii) $\{x \ll \xi\} \in \mathcal{R}$, $x \in L$.

(iv) $\{\xi \in F\} \in \mathcal{R}$, $F \in \mathcal{L}$.

<u>All these conditions imply</u>

(v) $\{\xi \leq x\} \in \mathcal{R}$, $x \in L$.

<u>Proof</u>: Let A and $\mathcal{A}$ be countable separating subsets of L and

L, resp. Fix $F \in L$. Clearly $F = \cup_{x \in F} \uparrow x$. But if $x \in F$ then $y \ll x$ for some $y \in F \cap A$. Hence

$$F = \cup \{\uparrow x;\ x \in F \cap A\}.$$

We see that (ii) implies (iv). Next fix $x \in L$. If $x \ll y$ then $y \in F$ for some $F \in A$ with $F \subseteq \uparrow x$. Therefore,

$$\{y;\ x \ll y\} = \cup \{F \in A;\ F \subseteq \uparrow x\}.$$

Moreover,

$$\uparrow x = \cap \{\{z;\ y \ll z\};\ y \in A,\ y \ll x\}.$$

Hence (iv) implies (iii) and (iii) implies (ii).

To see the final assertion, just note that $\downarrow x$ is Scott closed for all $x \in L$. QED

We have not been able to prove that condition (v) of Proposition 3.2 implies measurability of ξ. Indeed we believe that this is not possible without further restrictions on L. However a counterexample is lacking.

Simple sufficient conditions for Scott continuity are of interest to us. The next result will be applied to probability measures on continuous posets.

<u>PROPOSITION 3.3:</u> <u>Let</u> c <u>be a mapping between two continuous posets which have second countable Scott topologies</u>. <u>If</u> $c(x_n) \uparrow c(x)$ as $x_n \uparrow x$ <u>then</u> c <u>is Scott continuous</u>.

<u>Proof</u>: Let $\{x_n\} \subseteq L$ be countable and directed. Put $x = \vee_n x_n$. If $x \in \{x_n\}$ there is nothing to prove, so let us assume that $x \notin \{x_n\}$. Choose $n(1) > 1$ such that $x_1 \leq x_{n(1)}$. Then choose $n(2) > n(1)$ such that $x_i \leq x_{n(2)}$ for $1 \leq i \leq n(1)$. By continuing in this manner we obtain an increasing sequence $\{x_{n(k)}\}$ with join x. By

assumption

$$c(x)=\vee_k c(x_{n(k)})\leq\vee_n c(x_n)\leq c(x).$$

Thus we have equality throughout. Now Scott continuity is easy to prove. Cf [14]. QED

The following consequence needs no proof. Note that it can be extended to directed and countable collections of measurable functions.

COROLLARY 3.4: *Let* μ *be a measure, and let* $\{A_n\}$ *be a directed and countable collection of measurable sets. Then*

$$\mu\cup_n A_n=\vee_n \mu A_n.$$

4. Existence and uniqueness

Our first result discusses some continuity properties of probability measures on continuous posets. The uniqueness theorem then is a simple consequence. Finally we discuss necessary and sufficient conditions for the existence of probability measures on continuous semi-lattices.

By a random variable in a continuous poset L we understand a measurable mapping of some probability space, usually denoted $(\Omega,\mathcal{R},P)$, into L. The distribution of a random variable in L is the induced probability measure on (L,Σ). Let ξ,η be random variables in L. We write $\xi\overset{d}{=}\eta$ if the distributions of ξ and η coincide, i e if $P\xi^{-1}=P\eta^{-1}$.

PROPOSITION 4.1: Let ξ be a random variable in a continuous poset L. Suppose Scott(L) has a countable open base. Then, for all $F\in\mathcal{L}$,

$$P\{\xi\in F\}=\vee_{G\ll F}P\{\xi\in G\}=\vee_{x\in F}P\{x\leq\xi\}=\vee_{x\in F}P\{x\ll\xi\}.$$

Moreover, for each $x\in L$,

$$P\{x\leq\xi\}=\wedge_{y\ll x}P\{y\leq\xi\}=\wedge_{y\ll x}P\{y\ll\xi\}=\wedge_{x\in F\in\mathcal{L}}P\{\xi\in F\}.$$

Proof: Let $A\subseteq L$ be countable and separating. Fix $F\in\mathcal{L}$. We saw in the proof of Proposition 3.2 that $F=\cup\{\uparrow x;\ x\in F\cap A\}$. It is not hard to see that $F\cap A$ is filtered. By Corollary 3.4,

$$P\{\xi\in F\}=\vee_{x\in F\cap A}P\{x\leq\xi\}.$$

If $x\in F$ then $x\in G$ for some $G\in\mathcal{L}$ with $G\ll F$ and, moreover, $y\ll x$ for some $y\in F$. Now the first assertion of the proposition is obvious.

To see the second, first note that, for $x \in L$,

$$\uparrow x = \cap_{y \in A,\, y \ll x}\{z;\ y \ll z\} = \cap_{y \in A,\, y \ll x} \uparrow y,$$

where both intersections are filtered. By Corollary 3.4,

$$P\{x \le \xi\} = \wedge_{y \in A,\, y \ll x} P\{y \ll \xi\} = \wedge_{y \in A,\, y \ll x} P\{y \le \xi\}.$$

Moreover, if $y \ll x$ then $x \in F \subseteq \uparrow y$ for some $F \in \mathcal{L}$. Now the second and final assertion of the proposition follows at once. QED

By combining Proposition 4.1 with Proposition 3.2 we obtain the following uniqueness result.

THEOREM 4.2: Let ξ *and* η *be random variables in a continuous poset* L. *Suppose* L *is closed under finite non-empty joins or meets, and that* Scott(L) *has a countable open base. Then the following four statements are equivalent.*

(i) $\xi \overset{d}{=} \eta$;

(ii) $P\{x \ll \xi\} = P\{x \ll \eta\}$, $x \in L$;

(iii) $P\{x \le \xi\} = P\{x \le \eta\}$, $x \in L$;

(iv) $P\{\xi \in F\} = P\{\eta \in F\}$, $F \in \mathcal{L}$.

Proof: Of course (i) implies (ii). By Proposition 4.1, (ii) implies (iii) and (iii) implies (iv). If L is closed under finite non-empty meets, then $\mathcal{L} \cup \{\phi\}$ is closed under finite non-empty intersections and, therefore, (iv) implies (i). But Proposition 4.1 also shows that (ii) implies (iv) and (iv) implies (iii). If L is closed under finite non-empty joins, then the collection $\{\uparrow x;\ x \in L\}$ is closed under finite non-empty intersections, and, therefore, (iii) implies (i). QED

We continue to discuss our existence criteria for

probability measures on continuous semi-lattices. Suppose e is a real-valued mapping on some semi-lattice L. Let $y \in L$. Then we write $\Delta_y e$ for the mapping on L defined by putting

$$\Delta_y e(x) = e(x) - e(x \wedge y), \quad x \in L.$$

Cf [2]. Note that Δ_y may be regarded as an operator on the collection of all mappings from L to R. We will be particularly concerned with iterates of such operators.

Let $x, x_1, \ldots, x_n \in L$. A simple induction procedure yields

(4.1) $$\Delta_{x_1} \cdots \Delta_{x_n} e(x) = e(x) - \Sigma_i e(x \wedge x_i) + \Sigma_{i<j} e(x \wedge x_i \wedge x_j) - \ldots + (-1)^n e(x \wedge x_1 \wedge \ldots \wedge x_n).$$

Moreover, if $x \le x_i$ for some i, then

(4.2) $$\Delta_{x_1} \cdots \Delta_{x_n} e(x) = 0.$$

We conclude from (4.1) that the mapping $e \to \Delta_{x_1} \cdots \Delta_{x_n} e$ only depends on the finite set $A = \{x_1, \ldots, x_n\}$. Accordingly we sometimes write $\Delta_A = \Delta_{x_1} \cdots \Delta_{x_n}$. We also put $\Delta_\emptyset = e$.

We furthermore conclude from (4.1) that

(4.3) $$\Delta_A e(x) = \Delta_{A \wedge x} e(x),$$

where $A \wedge x = \{y \wedge x;\ y \in A\}$. Moreover, by (4.1) and (4.2) if $y \wedge x \le z$ for some distinct $y, z \in A$ then

(4.4) $$\Delta_A e(x) = \Delta_{A \setminus \{y\}} e(x).$$

In particular, if $y \le z$, $y \ne z$ then $\Delta_A = \Delta_{A \setminus \{y\}}$.

Now let us suppose that L is closed under finite non-empty joins (i e L^* is a semi-lattice), and let $\Lambda: L \to R$. We define

$$\Lambda_1(x; x_1) = \Lambda(x) - \Lambda(x \vee x_1), \quad x, x_1 \in L$$

and recursively for $n \ge 2$

$$\Lambda_n(x;x_1,\ldots,x_n)=\Lambda_{n-1}(x;x_1\ldots,x_{n-1})$$
$$-\Lambda_{n-1}(x\vee x_n;x_1,\ldots,x_{n-1}),\quad x,x_1,\ldots,x_n\in L.$$

Put $e(\uparrow x)=\Lambda(x)$, $x\in L$. Then, for all $n\in N$ and $x,x_1,\ldots,x_n\in L$,

(4.5) $$\Lambda_n(x;x_1,\ldots,x_n)=\Delta_{\uparrow x_1}\cdots\Delta_{\uparrow x_n}e(\uparrow x).$$

To see this, it is enough to note that

$$\uparrow x\cap\uparrow y=\uparrow x\vee y,\quad x,y\in L.$$

We may now state our existence theorems.

Theorem 4.3: Let L be a continuous semi-lattice with a top, and let $c:\mathcal{L}\to[0,1]$. Suppose Scott(L) has a countable open base. Then c extends to a unique probability measure on (L,Σ) iff

(i) $\Delta_{F_1}\cdots\Delta_{F_n}c(F)\geq 0$, $n\in N$, $F,F_1,\ldots,F_n\in\mathcal{L}$;

(ii) $c(F)=\lim_n c(F_n)$, $F,F_1,F_2,\ldots\in\mathcal{L}$, $F_n\uparrow F$;

(iii) $c(L)=1$.

We postpone the proof. In the special case when L is a continuous lattice we can say more. Let us write 0 for the bottom of L.

THEOREM 4.4: Let L be a continuous lattice and let $\Lambda:L\to[0,1]$. Suppose Scott(L) has a countable open base. Then there exists a unique probability measure λ on (L,Σ) satisfying

$$\lambda\uparrow x=\Lambda(x),\quad x\in L$$

iff

(i) $\Lambda_n(x;x_1,\ldots,x_n)\geq 0$, $n\in N$, $x,x_1,\ldots,x_n\in L$;

(ii) $\Lambda(x)=\lim_n\Lambda(x_n)$, $x,x_1,x_2,\ldots\in L$, $x_n\uparrow x$;

(iii) $\Lambda(0)=1$.

Before the proofs of these two existence theorems let us note that the latter can be weakened to continuous semi-lattices which are closed under finite non-empty joins provided condition (iii) is replaced by

(iii') $$\vee_{x\in L}\Lambda(x)=1.$$

To see this, suppose L is such a continuous poset. Add a bottom 0 to L and put $\Lambda(0)=1$. Check that the presumptions of Theorem 4.4 are at hand and conclude that there is a probability measure λ on $L\cup\{0\}$ satisfying $\lambda\uparrow x=\Lambda(x)$, $x\in L\cup\{0\}$. Then note that the family $\{\uparrow x;\ x\in L\}$ is directed. Its union is L. By Proposition 3.4 and (iii') we now get

$$\lambda\{0\}=1-\vee_{x\in L}\lambda\uparrow x=0.$$

That is, λ is concentrated on L.

Let us say that a mapping $\Lambda:L\to[0,1]$ is a <u>distribution function</u> if it satisfies the three conditions of Theorem 4.4. Our proofs of these theorems are very close to Matheron's proof of Choquet's original result [2][11]. It is given in a series of lemmata. Proofs are given only when required by the present higher generality.

Our first lemma discusses the necessity of condition (i) of Theorem 4.4. Its proof is left to the reader. The necessity of all the remaining conditions are obvious.

<u>LEMMA 4.5</u>: <u>Let</u> μ <u>be a probability measure on</u> L <u>and write</u> $M(x)=\mu\uparrow x$, $x\in L$. <u>Then</u>

$$M_n(x;x_1,\ldots,x_n)=\mu\uparrow x\backslash\cup_{i=1}^{n}\uparrow x_i,\quad n\in N,\quad x,x_1,\ldots,x_n\in L.$$

We proceed to discuss the sufficiency of the three conditions of Theorem 4.4. Recall that a collection S of subsets of L is called a <u>semi-ring</u>. If (i) $\emptyset\in S$, (ii) $S_1\cap S_2\in S$ whenever $S_1,S_2\in S$, and (iii) $S_1,S_2\in S$, $S_1\subseteq S_2$ imply that $S_2\backslash S_1$ is a union of a finite family of pairwise disjoint members of S. It is a <u>semi-field</u> if furthermore $L\in S$.

Let F be a collection of filters on L which is closed under finite non-empty intersections. Then F is a semi-lattice. We put

(4.6) $$S=\{A\backslash\cup A;\ A\in F,\ A\subseteq F\ \text{finite}\}.$$

<u>LEMMA 4.6: The collection S, defined in (4.6), is a semi-ring of subsets of L. It is a semi-field if $L\in F$.</u>

<u>LEMMA 4.7: Let $S\in S$ be non-empty. Then $S=B\backslash\cup_i A_i$ for some $B\in F$ and some finite $\{A_i\}\subseteq F$ satisfying</u>

(i) $A_i\subsetneq B$ for all i;

(ii) $A_1\not\subseteq A_2$ for all distinct $A_1,A_2\in\{A_i\}$.

Cf with (4.3) and (4.4). Any representation $B\backslash\cup A$ of a non-empty $S\in S$ satisfying the conclusion of Lemma 4.7 is called a <u>reduced representation</u>.

The next two lemmata need not be commented on in the case discussed by Choquet [2][11].

LEMMA 4.8: Let $B, C \in \mathcal{F}$ and let $\mathcal{A} \subseteq \mathcal{F}$ be finite. If $\emptyset \neq B \setminus \cup \mathcal{A} \subseteq C$ then $B \subseteq C$.

Proof: Let $x \in B$. We must prove that $x \in C$. If $x \notin \cup \mathcal{A}$ this is obvious, so let us assume that $x \in A$ for some $A \in \mathcal{A}$. Choose $y \in B \setminus \cup \mathcal{A}$. Then $x \wedge y \in B$. If $x \wedge y \in \cup A$ then $y \in \cup \mathcal{A}$. This is not true. Hence $x \wedge y \notin \cup \mathcal{A}$. Hence $x \wedge y \in C$, which implies $x \in C$. QED

LEMMA 4.9: Let $B \in \mathcal{F}$ and let $\mathcal{A} \subseteq \mathcal{F}$ be finite. If $B \subseteq \cup \mathcal{A}$ then $B \subseteq A$ for some $A \in \mathcal{A}$.

Proof: Since $B \neq \emptyset$, $\mathcal{A}$ must contain at least one filter. Suppose $B \not\subseteq A$ for all $A \in \mathcal{A}$. For every $A \in \mathcal{A}$ we then choose $x_A \in B \setminus A$. Since $\mathcal{A}$ is finite and non-empty, $x = \wedge_A x_A \in B$. But then $x \in A$, and therefore $x_A \in A$, for some $A \in \mathcal{A}$. A contradition, from which the lemma follows. QED

Now these two lemmata are used in the proof of the following uniqueness theorem for reduced representations.

LEMMA 4.10: Let $A \setminus \cup \mathcal{A}$ and $B \setminus \cup \mathcal{B}$ be two reduced representations of a non-empty member of $\mathcal{S}$. Then $A = B$ and $\mathcal{A} = \mathcal{B}$.

Proof: The conclusion $A = B$ follows at once from Lemma 4.8. But then we must have $\cup \mathcal{A} = \cup \mathcal{B}$ too. Fix $A' \in \mathcal{A}$. By Lemma 4.9, $A' \subseteq B'$ for some $B' \in \mathcal{B}$ and, moreover, $B' \subseteq A''$ for some $A'' \in \mathcal{A}$. But then $A' \subseteq A''$. Since the representations are reduced, $A' = A''$. We conclude that $\mathcal{A} \subseteq \mathcal{B}$. Of course the same argument may be applied to show $\mathcal{B} \subseteq \mathcal{A}$. QED

The next result follows at once from (4.3), (4.4) and

Lemma 4.10.

LEMMA 4.11: Let $e:\mathcal{F}\to R$. If $A\backslash\cup\mathcal{A}$ and $B\backslash\cup\mathcal{B}$ are two representations of a member of $\mathcal{S}$, then

$$\Delta_{\mathcal{A}}e(A)=\Delta_{\mathcal{B}}e(B).$$

Thus, whenever $e:\mathcal{F}\to R$ we may define a mapping λ on $\mathcal{S}$ by putting

$$\lambda A\backslash\cup\mathcal{A}=\Delta_{\mathcal{A}}e(A),\quad A\in\mathcal{F},\ \mathcal{A}\subseteq\mathcal{F}\ \text{finite.} \tag{4.7}$$

LEMMA 4.12: Let $e:\mathcal{F}\to R$. Then the mapping λ on $\mathcal{S}$, defined in (4.7), is additive.

Proof: Fix $S_1,S_2\in\mathcal{L}$ such that $S_1\cap S_2=\emptyset$ while $S_1\cup S_2=S\in\mathcal{S}$. Of course

$$\lambda S_1\cup S_2=\lambda S_1+\lambda S_2 \tag{4.8}$$

if S_1 or S_2 is empty, so let us assume both to be non-empty. We further assume that $A\backslash\cup\mathcal{A}$, $B\backslash\cup\mathcal{B}$ and $C\backslash\cup\mathcal{C}$ are reduced representations of S_1,S_2 and S, resp.

By Lemma 4.8, $A\cup B\subseteq C$. Clearly $A\cap B\subseteq\cup(\mathcal{A}\cup\mathcal{B})$. By Lemma 4.9, $A\cap B$ is included in some member of $\mathcal{A}\ \mathcal{B}$. Let us assume

$$A\cap B\subseteq A', \tag{4.9}$$

where $A'\in\mathcal{A}$, and prove that this implies $A=C$. (Analogously the reader may show that $B=C$ follows from $A\cap B\subseteq\cup\mathcal{B}$.)

If $C\subseteq B$ then $A\subseteq B$ and $A\subseteq A'$ follows by (4.9). This is clearly impossible. Hence we may select a point $x\in C\backslash B$. In order to obtain yet another contradition, suppose there is a point $y\in C\backslash A$. Then $x\wedge y\in C$. Let $\{C_i\}$ be an enumeration of $\mathcal{C}$, and choose for each i some $x_i\in C\backslash C_i$. Then

$$z=x\wedge y\wedge\wedge_i x_i\in C\setminus\cup\mathcal{C},$$

since $z\in C_i$ implies $x_i\in C_i$. It follows that $z\in A\cup B$. This leads at once to a contradiction and we conclude that $C\subseteq A$. Hence $A=C$ as claimed in the above paragraph.

But then $B\subseteq A$ and from (4.9) we get $B\subseteq A'$. Let $x\in S$. Then $x\in C$. If $x\in L\setminus A'$ then $x\in B^c$ and, therefore $x\in S_1$. Thus we have $S\setminus A'\subseteq S_1$. The reverse inclusion is obvious. Moreover, if $x\in A'$ then $x\in S_1$ and we must have $x\in S_2$. Hence $S\cap A'\subseteq S_2$. Again the reverse inequality is obvious. We thus have

$$S_1=S\setminus A'=C\setminus\cup(\mathcal{C}\cup\{A'\});$$

$$S_2=S\cap A'=C\cap A'\setminus\cup\mathcal{C}$$

and see

$$\lambda S_1+\lambda S_2=\Delta_{\mathcal{C}}\Delta_A\cdot e(C)+\Delta_{\mathcal{C}}e(C\cap A')=\Delta_{\mathcal{C}}e(C)=\lambda S.$$

Thus (4.8) holds if (4.9) is at hand. The remaining case is completely similar. QED

Below we will prove, for a suitable choice of $\mathcal{F}$ and e, that λ is both non-negative and continuous. For this we need the following result, which needs no proof in the case discussed by Choquet [2][11].

<u>LEMMA 4.13:</u> <u>Fix</u> $x\in L$ <u>and</u> $F\in\mathcal{L}$. <u>Then there are sequences</u> $\{y_n\},\{z_n\}\subseteq L$ <u>and</u> $\{G_n\},\{H_n\}\in\mathcal{L}$ <u>such that</u>

$$G_1\supseteq\uparrow y_1\supseteq G_2\supseteq\ldots\supseteq\uparrow x=\cap_n G_n=\cap_n\uparrow y_n;$$

$$\uparrow z_1\subseteq H_1\subseteq\uparrow z_2\subseteq\ldots\subseteq F=\cup_n\uparrow z_n=\cup_n H_n.$$

<u>Proof</u>: Let $A\subseteq L$ be countable and separating. Then the set $\{y\in A;\ y\ll x\}$ is countable and directed. Its join equals x. Now proceed as in the proof of Proposition 3.3 and conclude that

there is a sequence $\{y_n\} \subseteq L$, satisfying $y_n \ll y_{n+1} \ll x$ for all n, with join x. Then choose $\{G_n\} \subseteq \mathcal{L}$ such that $\uparrow y_{n+1} \subseteq G_{n+1} \subseteq \uparrow y_n$. The reader easily shows that $\uparrow x = \cap_n \uparrow y_n = \cap_n G_n$, thereby completing the proof of the first part of the lemma.

To see the second part, proceed as above and conclude that there is a sequence $\{H_n\} \subseteq \mathcal{L}$ fulfilling $H_n \ll H_{n+1} \ll F$, $n \in N$, and $\vee_n H_n = F$. Then choose $\{z_n\} \subseteq L$ such that $H_n \subseteq \uparrow z_{n+1} \subseteq H_{n+1}$.
QED

We now fix

$$e(A) = \vee_{x \in A} \lambda(x), \quad A \in \mathcal{F} \tag{4.10}$$

and define λ as in (4.7). The collection $\mathcal{F}$ will soon be specified.

In order to complete our proof of Theorem 4.4 we need to know that e satisfies certain continuity properties. Hence the Lemmata 4.14-4.18.

<u>LEMMA 4.14:</u> <u>Whenever</u> $A, A_1, A_2 \in \mathcal{F}$ <u>we have</u>

$$e(A \cap A_1) \le e(A);$$

$$e(A \cap A_1) + e(A \cap A_2) \le e(A) + e(A \cap A_1 \cap A_2).$$

<u>Proof:</u> The first assertion is equivalent to saying that e is increasing. It needs no proof. Now choose $x_1 \in A \cap A_1$ and $x_2 \in A \cap A_2$. Then $x = x_1 \wedge x_2 \in A$ and $x_1 \vee x_2 \in A \cap A_1 \cap A_2$. Hence

$$\lambda(x_1) + \lambda(x_2) = \lambda(x \vee x_1) + \lambda(x \vee x_2)$$

$$\le \lambda(x) + \lambda(x \vee x_1 \vee x_2) = \lambda(x) + \lambda(x_1 \vee x_2)$$

$$\le e(A) + e(A \cap A_1 \cap A_2).$$

Now the second assertion of the lemma is obvious. QED

LEMMA 4.15: *Let* $A_i, B_i \in F$, $i=1,2,\ldots$, *and suppose* $A_i \subseteq B_i$ *for each* i. *Then, for every* $n=1,2,\ldots$,

$$e(\cap_{i=1}^{n} B_i)+\Sigma_{i=1}^{n} e(A_i) \le e(\cap_{i=1}^{n} A_i)+\Sigma_{i=1}^{n} e(B_i).$$

Proof: First suppose $n=2$. (The result is obvious if $n=1$.) Put $D=B_1$, $D_1=A_1$ and $D_2=A_2$. By Lemma 4.14,

$$e(A_1)+e(A_2 \cap B_1) \le e(B_1)+e(A_1 \cap A_2).$$

Next put $D=B_2$, $D_1=B_1$ and $D_2=A_2$. Then

$$e(B_1 \cap B_2)+e(A_2) \le e(B_2)+e(A_2 \cap B_1).$$

Add these expressions and cancel $e(A_2 \cap B_1)$ from both sides.

Now suppose the result is true whenever $n \le m$, $m \ge 2$. Then

$$e(\cap_{i=1}^{m+1} B_i)+e(\cap_{i=1}^{m} A_i)+e(A_{m+1})$$

$$\le e(\cap_{i+1}^{m+1} A_i)+e(\cap_{i=1}^{m} B_i)+e(B_{m+1})$$

Add $\Sigma_{i=1}^{m} e(A_i)$ to both sides of this inequality and use the supposition above:

$$e(\cap_{i=1}^{m+1} B_i)+\Sigma_{i=1}^{m+1} e(A_i)+e(\cap_{i=1}^{m} A_i)$$

$$\le e(\cap_{i=1}^{m+1} A_i)+\Sigma_{i=1}^{m+1} e(B_i)+e(\cap_{i=1}^{m} A_i).$$

Thus the result is true for $n=m+1$. By induction the lemma follows. QED

The next result presumes $\uparrow x \in F$, $x \in L$. Note that then

$$e(\uparrow x)=\Lambda(x), \quad x \in L.$$

LEMMA 4.16: *Suppose* $\uparrow x \in F$, $x \in L$. *Let* $A, A_1, A_2, \ldots \in F$ *and suppose* $A_n \downarrow A$. *Then*

$$e(A)=\lim_n e(A_n).$$

Proof: For $n=1,2,\ldots$ choose $x_n \epsilon A_n$ with

$$0 \le e(A_n) - \Lambda(x_n) \le \epsilon \cdot 2^{-n}$$

for some fixed $\epsilon>0$. Note that $\uparrow x_n \subseteq A_n$. Hence, by Lemma 4.15,

$$0 \le e(A_n) - \Lambda(\vee_{i=1}^{n} x_i) \le \epsilon$$

We now see that

$$e(A) \le \lim_n e(A_n) = \lim_n \Lambda(\vee_{i=1}^{n} x_i) = \Lambda(\vee_i x_i)$$

Note that

$$\uparrow \vee_i x_i = \cap_i \uparrow x_i \subseteq \cap_i A_i = A.$$

Hence $\Lambda(\vee_i x_i) \le e(A)$. QED

The following two lemmata require that $\uparrow x \cap F \epsilon \mathcal{F}$ for all $x \epsilon L$ and $F \epsilon \mathcal{L}$. Of course this implies that $L \epsilon \mathcal{F}$ and that $\uparrow x \epsilon \mathcal{F}$, $x \epsilon L$. Thus Lemma 4.16 is at our disposal.

LEMMA 4.17: Suppose $\uparrow x \cap F \subseteq \mathcal{F}$, $x \epsilon L$, $F \epsilon \mathcal{L}$. Let $x \epsilon L$, $F \epsilon \mathcal{L}$ and let $\{y_n\} \subseteq L$. Suppose $(\uparrow y_n) \uparrow F$. Then

$$e(\uparrow x \cap F) = \lim_n e(\uparrow x \vee y_n).$$

Proof: Since $x \vee y_n \epsilon \uparrow x \cap F$,

$$\vee_n \Lambda(x \vee y_n) \le e(\uparrow x \cap F).$$

But if $y \epsilon \uparrow x \cap F$ then $x \vee y_n \le y$ for sufficiently large n. Hence

$$\Lambda(y) \le \vee_n \Lambda(x \vee y_n),$$

from which

$$e(\uparrow x \cap F) \le \vee_n \Lambda(x \vee y_n)$$

follows. QED

LEMMA 4.18: Suppose $\uparrow x \cap F \epsilon \mathcal{F}$, $x \epsilon L$, $F \epsilon \mathcal{L}$. Fix $x, y \epsilon L$ and $F \epsilon \mathcal{L}$.

Let further $\{z_n\} \subseteq L$ and $\{G_n\} \subseteq \mathcal{L}$. Suppose $(\uparrow z_n) \uparrow F$ while $G_n \downarrow (\uparrow x)$. Then

$$e(\uparrow x \vee y \cap F) = \lim_n e(\uparrow y \vee z_n \cap G_n).$$

Proof: Let us first note that

$$x \vee y \vee z_n \in \uparrow y \vee z_n \cap G_n \subseteq \uparrow y \cap F \cap G_n.$$

Hence

$$e(\uparrow x \vee y \vee z_n) \le e(\uparrow y \vee z_n \cap G_n) \le e(\uparrow y \cap F \cap G_n).$$

By Lemma 4.17,

$$\lim_n e(\uparrow x \vee y \vee z_n) = e(\uparrow x \vee y \cap F)$$

and by Lemma 4.16,

$$\lim_n e(\uparrow y \cap F \cap G_n) = e(\uparrow x \vee y \cap F).$$

Now the conclusion of the lemma is obvious. QED

Let us now fix our collection $\mathcal{F}$ to be

(4.11) $$\mathcal{F} = \{\uparrow x \cap F;\ x \in L,\ F \in \mathcal{L}\}.$$

Of course this is a semi-lattice. The next result allows the conclusion that λ (see (4.7)) is an additive mapping of $\mathcal{S}$ into $[0,\infty)$.

LEMMA 4.19: Let $A \in \mathcal{F}$ and let $\mathcal{A} \subseteq \mathcal{F}$ be finite. Then

$$\Delta_{\mathcal{A}} e(A) \ge 0.$$

Proof: Suppose $\mathcal{A} = \{\uparrow x_i \cap F_i;\ 1 \le i \le m\}$ and $A = \uparrow x_0 \cap F_0$. For $0 \le i \le m$ choose some sequence $\{y_{in}\} \subseteq L$ such that $(\uparrow y_{in}) \uparrow F$. Suppose $0 \in I \subseteq \{0,\dots,m\}$. Then, writing $y_{In} = \vee_{i \in I} y_{in}$,

$$\uparrow y_{I1} \subseteq \uparrow y_{I2} \subseteq \dots \subseteq \cap_{i \in I} F_i = \cap_n \uparrow y_{In}.$$

Now $\Delta_{\mathcal{A}} e(A) \ge 0$ follows by (4.1) and Lemma 4.17. QED

Introduce

$$\mathcal{C}=\{\uparrow x\backslash\cup\mathcal{A};\ x\in L,\ \mathcal{A}\subseteq\mathcal{L}\ \text{finite}\}.$$

Note that $\mathcal{C}$ is closed under non-empty finite intersections. Moreover, in the terminology of [12], $\mathcal{C}$ is a <u>compact</u> class of subsets of L. This means that, whenever $C_n\downarrow\emptyset$, where $C_1,C_2,\ldots\in\mathcal{C}$, we have $C_n=\emptyset$ for n sufficiently large. To see this, suppose $C_n=\uparrow x_n\backslash\cup\mathcal{A}_n$. Then $\cap_n C_n=\emptyset$ only if $\uparrow\vee_n x_n\subseteq\cup_n(\cup\mathcal{A}_n)$. But $\uparrow\vee_n x_n$ is Scott compact. Hence $\uparrow\vee_n x_n\subseteq\cup_{n\leq m}(\cup\mathcal{A}_n)$ for some m. Our claim now comes from the fact that $\cup_{n\leq m}(\cup\mathcal{A}_n)$ is Scott open.

Suppose $\emptyset\neq S\in\mathcal{S}$. Then

$$S=\uparrow x\cap F\backslash\cup\mathcal{A},$$

where $\mathcal{A}=\{\uparrow x_i\cap F_i;\ 1\leq i\leq m\}$. Choose sequences $\{y_n\}\subseteq L$ and $\{G_{in}\}\subseteq\mathcal{L}$, $1\leq i\leq m$, such that $(\uparrow y_n)\uparrow F$ while $G_{in}\downarrow(\uparrow x_i)$. Write

$$C_n=\uparrow x\vee y_n\backslash\cup\{G_{in}\cap F_i;\ 1\leq i\leq m\}$$

Then $C_n\in\mathcal{C}$ and, by (4.1) and the Lemmata 4.17 and 4.18,

$$\lambda S=\lim_n\lambda C_n.$$

Hence

$$\lambda S=\vee\{\lambda C;\ C\in\mathcal{C},\ C\subseteq S\},\ S\in\mathcal{L}.$$

Note that $\lambda L=\lambda\uparrow 0=\Lambda(0)=1$. By Proposition I.6.2 of [12] it now follows that λ extends to a probability measure on (L,Σ). This completes our proof of Theorem 4.4

We continue to discuss the proof of Theorem 4.3. Suppose L is a continuous semi-lattice with a top having a second countable Scott topology and let $c:\mathcal{L}\to[0,1]$ satisfy the three conditions of this theorem. Whenever $U\in\text{Scott}(L)$ we write

$$\Lambda(U)=\wedge\{c(\cap_{i=1}^{n}F_i);\ n\in N,\ F_1,\ldots,F_n\in U\}.$$

Our aim is to show that Λ is the distribution function of some

probability measure λ on Scott(L). Here we will use the already proved Theorem 4.4. Then we show that λ concentrates its mass to OFilt(L). Our final argumentation uses the Lawson duality.

First note that condition (iii) of Theorem 4.4 trivially holds. Clearly Λ is decreasing. Thus if $U_n \uparrow U$ then

$$\Lambda(U) \leq \wedge_n \Lambda(U_n) = \lim_n \Lambda(U_n).$$

Suppose $\Lambda(U) < x$. Then there exist some $F_1, \ldots, F_m \in U$ with $c(\cap_{i=1}^m F_i) \leq x$. But then

$$\cup_{i=1}^m \uparrow F_i \subseteq U = \cup_n U_n.$$

Since $\cup_{i=1}^m \uparrow F_i$ is a Scott compact subset of L, we must have $\cup_{i=1}^m \uparrow F_i \subseteq U_n$, and therefore $F_1, \ldots, F_m \in U_n$, for n sufficiently large. But then $\wedge_n \Lambda(U_n) \leq c(\cap_{i=1}^m F_i) \leq x$. Hence

$$\Lambda(U) = \lim_n \Lambda(U_n).$$

That is, condition (ii) of Theorem 4.4 is at hand too.

Fix $k \in N$ and $U_0, U_1, \ldots, U_k \in$ Scott(L). For $j=0,1,\ldots,k$ and $n \in N$ choose $F_{jn1}, \ldots, F_{jnm(n)} \in L$ such that

$$(H_{jn})^0 \uparrow U_j, \quad j=0,1,\ldots,k,$$

where

$$H_{jn} = \cup_{i=1}^{m(n)} \uparrow F_{jni}, \quad j=0,1,\ldots,k, \ n \in N.$$

Let $0 \in J \subseteq \{0,1,\ldots,k\}$, and write

$$H_{Jn} = \cup_{j \in J} H_{jn}, \quad n \in N.$$

Then, for all such J's,

$$H_{J1} \subseteq H_{J2} \subseteq \ldots \subseteq \cup_{j \in J} U_j = \cup_n H_{Jn}.$$

It is not hard to see that

$$\lim_n c(\cap_{j \in J} \cap_{i=1}^{m(n)} F_{jni}) = \Lambda(\cup_{j \in J} U_j)$$

and, therefore,

$$\Lambda_k(U;U_1,\ldots,U_k)\geq 0.$$

Thus, also condition (i) of Theorem 4.4 is at hand.

We may now conclude that there is a probability measure λ on Scott(L) with distribution function Λ. This concludes the first part of our proof of Theorem 4.3.

We proceed to prove that

(4.12) $$\lambda \text{OFilt}(L) = 1.$$

The following result is useful.

PROPOSITION 4.20: Let η be a random variable in Scott(L), where L is a continuous semi-lattice. Suppose Scott(L) is second countable. Then $\eta\in$OFilt(L) a s iff

$$P\cap_{i=1}^{n}\{F_i\in\eta\}=P\{\cap_{i=1}^{n}F_i\in\eta\},\quad n\in N,\quad F_1,\ldots,F_n\in L$$

Proof: The necessity is obvious. To see the sufficiency, first note that it implies

$$P\{F_1,\ldots,F_n\in\eta,\ \cap_{i=1}^{n}F_i\notin\eta\}=0,\quad n\in N,\quad F_1,\ldots,F_n\in L.$$

Let $A\subseteq L$ be countable and separating. It is not hard to see that the probability of the even that

(4.13) $$F_1,\ldots,F_n\in\eta \text{ implies } \cap_{i=1}^{n}F_i\in\eta,\quad n\in N,\quad F_1,\ldots,F_n\in A$$

is one. By a straightforward approximation procedure it may be seen that A can be replaced by L in (4.13). This shows that $\eta\in$OFilt L a s. QED

Thus we must provide a proof of

(4.14) $$\lambda\{U;\ \cup_{i=1}^{n}\uparrow F_i\subseteq U\}=\lambda\{U;\ \uparrow\cap_{i=1}^{n}F_i\subseteq U\},\quad n\in N,\quad F_1,\ldots,F_n\in L.$$

Note that the sets $\{U;\ \cup_{i=1}^{n}\uparrow F_i \subseteq U\}$ and $\{U;\ \uparrow\cap_{i=1}^{n} F_i \subseteq U\}$ are open filters of Scott(L). Hence, by Proposition 4.1, and some straightforward argumentation,

$$\lambda\{U;\ \cup_{i=1}^{n}\uparrow F_i \subseteq U\} = \vee\{\Lambda(U);\ \cup_{i=1}^{n}\uparrow F_i \subseteq U\}$$

$$= \vee\{\Lambda(U);\ F_1,\dots,F_n \in U\} = c(\cap_{i=1}^{n} F_i)$$

and

$$\lambda\{U;\ \uparrow\cap_{i=1}^{n} F_i \subseteq U\} = \vee\{\Lambda(U);\ \cap_{i=1}^{n} F_i \in U\} = c(\cap_{i=1}^{n} F_i).$$

Thus (4.14) is at hand and we conclude by Proposition 4.20 that (4.12) holds. This shows the second step of our proof of Theorem 4.3.

Let $x \in L$. Then $F_x = \{F \in L;\ x \in F\} \in \mathrm{OFilt}(L)$. We get

$$\lambda\uparrow F_x = \Lambda(F_x) = \wedge\{c(\cap_{i=1}^{n} F_i);\ n \in N,\ F_1,\dots,F_n \in F_x\}$$

$$= \wedge\{c(\cap_{i=1}^{n} F_i);\ n \in N,\ \cap_{i=1}^{n} F_i \in F_x\} = \wedge_{x \in F} c(F).$$

By the Lawson duality (see (2.4)) we may regard λ as a measure on L. By doing so we obviously have

$$\lambda\uparrow x = \wedge_{x \in F} c(F), \quad x \in L.$$

It is now easily seen that

$$\lambda F = c(F), \quad F \in L.$$

This completes our proof of Theorem 4.3.

5. Convergence

In this section we will discuss weak convergence of probability measures on continuous semi-lattices w r t the so called Lawson topology. We first provide our continuous posets with this topology.

Let L be a continuous poset. By the Lawson topology on L we understand the topology generated by the open filters of L and the collection $L \setminus \uparrow x$, $x \in L$. It is known [8] that this topolgoy is completely regular and Hausdorff. If Scott(L) is second countable then so is the Lawson topology (see Proposition 3.1), and we may conclude from well-known results that L is Polish (i e a completely metrizable second countable space) (see e g [15]). In this case Σ coincides with the Borel-σ-field w r t the Lawson topology.

The paper [8] also proves that the Lawson topology is locally compact if L is closed under finite non-empty joins. In this case the filter $\uparrow x$ is easily seen to be compact for each $x \in L$.

Let us also note here that, if L is a semi-lattice, then the mapping $(x,y) \to x \wedge y$ is continuous in this topology.

Some of the facts quoted above follow rather easily from the following result, which is a cornerstone in our development.

PROPOSITION 5.1: Let L be a continuous poset. Suppose Scott(L) is second countable. Then the Lawson topology on OFilt(L) coincides with the restriction to this set of the Lawson topology on Scott(L).

<u>Proof</u>: First note that, for $x \in L$,

$$\{F \in \mathcal{L};\ x \in F\} = \{U \in \text{Scott}(L);\ \uparrow x \subseteq U\} \cap \mathcal{L}$$

while, for $H \in \mathcal{L}$,

$$\{F \in \mathcal{L};\ H \subseteq F\} = \{U \in \text{Scott}(L);\ H \subseteq U\} \cap \mathcal{L}.$$

Hence the Lawson topology on $\mathcal{L}$ is included in the relative Lawson topology.

To see the converse, let $K \subseteq L$ be compact and saturated w r t Scott(L). Then

$$K = K^s = \cap \{U \in \text{Scott}(L);\ K \subseteq U\}.$$

Since Scott(L) has a countable open base, $K = \cap_n U_n$ for some decreasing sequence $\{U_n\} \subseteq \text{Scott}(L)$. Note that if $K \subseteq U_n$ then $K \subseteq V_n$ for some $V_n \ll U_n$. Now choose for each n some finitely many $x_{n1}, \ldots, x_{nm(n)} \in L$ such that $V_n \subseteq \cup_i \uparrow x_{ni} \subseteq U_n$. Note that $K_n = \cup_i \uparrow x_n$ is compact and saturated. Clearly $K_n \downarrow K$. Hence

$$\{U \in \text{Scott}(L);\ K \subseteq U\} \cap \mathcal{L} = \cup_n \{F \in \mathcal{L};\ \cup_i \uparrow x_{ni} \subseteq F\} = \cup_n \cap_i \{F \in \mathcal{L};\ x_{ni} \in F\}.$$

Let further $V \in \text{Scott}(L)$. Then $V = \cup_\alpha F_\alpha$ for some $\{F_\alpha\} \subseteq \mathcal{L}$. We get

$$\{U \in \text{Scott}(L);\ V \not\subseteq U\} \cap \mathcal{L} = \cup_\alpha \{F \in \mathcal{L};\ F_\alpha \not\subseteq F\}.$$

Hence the relative topology on $\mathcal{L}$ is included in the Lawson topology. QED

Let $\xi, \xi_1, \xi_2, \ldots$ be random variables in L. We write $\xi_n \overset{d}{\to} \xi$ when ξ_n converges in distribution to ξ, i e when $P\xi_n^{-1}$ converges weakly to $P\xi^{-1}$. Cf [1].

We now state and prove the main result of this section.

<u>THEOREM 5.2</u>: <u>Let</u> $\xi, \xi_1, \xi_2, \ldots$ <u>be random variables in a continuous semi-lattice</u> L. <u>Suppose</u> Scott(L) <u>has a countable</u>

<u>open base</u>. <u>Then the following three conditions are equivalent</u>:

(i) $\xi_n \xrightarrow{d} \xi$;

(ii) $\begin{cases} \liminf_n P\{\xi_n \in F\} \geq P\{\xi \in F\}, & F \in \mathcal{L}, \\ \limsup_n P\bigcap_{i=1}^{m}\{x_i \leq \xi_n\} \leq P\bigcap_{i=1}^{m}\{x_i \leq \xi\}, & m \in N,\ x_1,\ldots,x_m \in L; \end{cases}$

(iii) $\begin{cases} \liminf_n P\{x \ll \xi_n\} \geq P\{x \ll \xi\}, & x \in L, \\ \limsup_n P\bigcap_{i=1}^{m}\{x_i \leq \xi_n\} \leq P\bigcap_{i=1}^{m}\{x_i \leq \xi\}, & m \in N,\ x_1,\ldots,x_m \in L; \end{cases}$

Before the proof, let us just note that the second part of conditions (ii) and (iii) reduces to

$$\limsup_n P\{x \leq \xi_n\} \leq P\{x \leq \xi\},\quad x \in L$$

if L is closed under finite non-empty joins.

<u>Proof:</u> Suppose (i). Then (iii) follows from the fact that $\uparrow x$ is closed while $\{y;\ x \ll y\}$ is open for each $x \in L$. Cf [1]. To see that (iii) implies (ii), note first that

$$P\{\xi \in F\} = \vee_{x \in F} P\{x \ll \xi\}, \qquad F \in \mathcal{L}.$$

Cf Proposition 4.1. Now fix $F \in \mathcal{L}$ and let $\epsilon > 0$. Then

$$P\{\xi \in F\} \leq P\{x \ll \xi\} + \epsilon$$

for some $x \in F$. Hence, by the first inequality of (iii),

$$\begin{aligned} P\{\xi \in F\} &\leq \liminf_n P\{x \ll \xi_n\} + \epsilon \\ &\leq \liminf_n P\{\xi_n \in F\} + \epsilon. \end{aligned}$$

Thus (iii) implies (ii) indeed.

To see that (ii) implies (i) we first consider the case when L is a continuous lattice. Then L is compact, which implies that (ξ_n) is relatively compact w r t convergence in distribution.

Select a subsequence $(\xi_{n(k)})$ and a random variable η

such that $\xi_{n(k)} \xrightarrow{d} \eta$. Then, since (i) implies (ii),

$$\begin{cases} \liminf_k P\{\xi_{n(k)} \in F\} \geq P\{\eta \in F\}, & F \in \mathcal{L}, \\ \limsup_k P\{x \leq \xi_{n(k)}\} \leq P\{x \leq \eta\}, & x \in L. \end{cases}$$

Let $\mathcal{A} \subseteq \mathcal{L}$ be countable and separating. It is easily seen that $\uparrow x = \cap_{x \in G \in \mathcal{A}} G$. Now suppose $x \in G \in \mathcal{A}$, and choose $y \in L$ and $F \in \mathcal{L}$ such that $\uparrow x \subseteq F \subseteq \uparrow y \subseteq G$. Then, by (ii),

$$\begin{aligned} P\{\xi \in G\} &\geq P\{y \leq \xi\} \geq \limsup_n P\{y \leq \xi_n\} \\ &\geq \limsup_n P\{\xi_n \in F\} \geq \liminf_k P\{\xi_{n(k)} \in F\} \\ &\geq P\{\eta \in F\} \geq P\{x \leq \eta\}. \end{aligned}$$

The collection $\{G \in \mathcal{A};\ x \in G\}$ is filtering. By Corollary 3.4,

$$P\{x \leq \xi\} \geq P\{x \leq \eta\}, \qquad x \in L.$$

Next fix $F \in \mathcal{L}$. If $G \in \mathcal{A}$, $G \ll F$ then $G \subseteq \uparrow x \subseteq F$ for some $x \in L$. Hence

$$P\{\xi \in F\} \geq P\{x \leq \xi\} \geq P\{x \leq \eta\} \geq P\{\eta \in G\}.$$

By an other reference to Corollary 3.4,

$$P\{\xi \in F\} \geq P\{\eta \in F\}, \qquad F \in \mathcal{L}.$$

Moreover, by (ii),

$$\begin{aligned} P\{\xi \in G\} &\leq \liminf_n P\{\xi_n \in G\} \leq \liminf_n P\{x \leq \xi_n\} \\ &\leq \limsup_k P\{x \leq \xi_{n(k)}\} \leq P\{x \leq \eta\} \leq P\{\eta \in F\}, \end{aligned}$$

and by yet another reference to Corollary 3.4,

$$P\{\xi \in F\} \leq P\{\eta \in F\}, \qquad F \in \mathcal{L}.$$

Collect the pieces, refer to Theorem 4.2 and conclude that $\xi \overset{d}{=} \eta$. By [1], $\xi_n \xrightarrow{d} \xi$ follows.

Let us now remove the extra assumption on L. For $x \in L$ write $\mathcal{F}(x) = \{F \in \mathcal{L};\ x \in F\}$. Let $K \subseteq L$ be Scott compact and saturated. We saw in the proof of Proposition 5.1 that there exist a sequence $\{\cup_{i=1}^{n(m)} \uparrow F_{mi}\}$ with

$$(\cup_{i=1}^{n(m)} \uparrow F_{mi}) \downarrow K.$$

Here the F_{mi}'s belong to L. It is not so hard to see that, by (ii),

$$\liminf_n P\{K \subseteq F(\xi_n)\} \geq \liminf_n P\{\cup_i \uparrow F_{mi} \subseteq F(\xi_n)\}$$
$$= \liminf_n P\{\xi_n \in \cap_i F_{mi}\} \geq P\{\xi \in \cap_i F_{mi}\}$$
$$= P\{\cup_i \uparrow F_{mi} \subseteq F(\xi)\} \uparrow P\{K \subseteq F(\xi)\}.$$

Hence, whenever $K \subseteq L$ is Scott compact and saturated,

$$\liminf_n P\{K \subseteq F(\xi_n)\} \geq P\{K \subseteq F(\xi)\}.$$

Next fix $U \in \mathrm{Scott}(L)$. Then $U = \cup_m F(x_m)$ for some countable $\{x_m\} \subseteq L$. By (ii) we now get

$$\limsup_n P\{U \subseteq F(\xi_n)\} = \limsup_n P \cap_m \{F(x_m) \subseteq F(\xi_n)\}$$
$$\leq \limsup_n P \cap_{k \leq m} \{x_k \leq \xi_n\}$$
$$\leq P \cap_{k \leq m} \{x_k \leq \xi\} \downarrow P \cap_m \{x_m \leq \xi\} = P\{U \subseteq F(\xi)\}.$$

Hence, whenever $U \in \mathrm{Scott}(L)$,

$$\limsup_n P\{U \subseteq F(\xi_n)\} \leq P\{U \subseteq F(\xi)\}.$$

We may now conclude by the already proved special case of the theorem that $F(\xi_n) \xrightarrow{d} F(\xi)$ w r t the Lawson topology on $\mathrm{Scott}(L)$. By Proposition 5.1, this implies that $F(\xi_n) \xrightarrow{d} F(\xi)$ w r t the Lawson topology on $\mathrm{OFilt}(L)$. By the Lawson duality we now see that $\xi_n \xrightarrow{d} \xi$.
QED

The next result shows that the collection of all pairs $(x,F) \in L \times L$ satisfying $F \subseteq \uparrow x$ and $P\{\xi \in \uparrow x \setminus F\} = 0$ is sufficiently rich for many purposes (cf condition (v) of Proposition 3.1).

<u>LEMMA 5.3:</u> <u>Suppose</u> ξ <u>is a random variable in a continuous semi-lattice</u> L, <u>which has a second countable Scott topology</u>.

Then, whenever $x \in F$, we have $x \in H \subseteq \uparrow z \subseteq F$ for some pair $(z,H) \in L \times \mathcal{L}$ satisfying $P\{z \leq \xi\} = P\{\xi \in H\}$.

Proof: Put $x(1)=x$ and choose $x(0) \in F$ such that $x(0) \ll x(1)$. Then choose $\{x(k \cdot 2^{-n});\ n \in N,\ k=1,2,\ldots,2^n-1\}$ such that $x(k \cdot 2^{-n}) \ll x(l \cdot 2^{-m})$ whenever $k \cdot 2^{-n} < l \cdot 2^{-m}$. Now put $y(t) = \vee\{x(k \cdot 2^{-n});\ k \cdot 2^{-n} < t\}$, $0<t<1$. It is easily seen that

$$y(t) = \vee_{s<t}\, y(s), \qquad 0<t<1.$$

Hence the mapping

$$f(t) = P\{y(t) \leq \xi\}, \qquad 0<t<1$$

is decreasing and left continuous. Let t be a point of continuity. Then, by monotone convergence, writing $z=y(t)$ and $H = \cup_{s>t} \uparrow y(s)$,

$$P\{z \leq \xi\} = P\{\xi \in H\}.$$

The reader easily verifies that $H \in \mathcal{L}$ and that $x \in H \subseteq \uparrow z \subseteq F$. QED

This leads directly to the following result, which supplements Theorem 5.2.

PROPOSITION 5.4: Let $\{\xi_n\}$ be a sequence of random variables in a continuous semi-lattice L which is closed under finite non-empty joins. Suppose Scott(L) is second countable. If ξ_n converges in distribution to some random variable ξ, then there are separating subsets A and $\mathcal{A}$ of L and $\mathcal{L}$, resp, with

(5.1) $$P\{x \leq \xi\} = \lim_n P\{x \leq \xi_n\}, \qquad x \in A;$$

(5.2) $$P\{\xi \in F\} = \lim_n P\{\xi_n \in F\}, \qquad F \in \mathcal{A}.$$

Let $\mathcal{B} \in \Sigma$ be such that, whenever $x \in F$, we have $\uparrow x \subseteq B \subseteq F$ for some $B \in \mathcal{B}$ and let $c: \mathcal{B} \to [0,1]$ be such that, whenever $\epsilon > 0$ we have

$c(B) \geq 1-\epsilon$ _for some_ $B \epsilon \mathcal{B}$ _having a lower bound_. (_This restriction on_ c _is superfluous if_ L _is a continuous lattice_.) _If_

(5.3) $$c(B)=\lim_n P\{\xi_n \epsilon B\}, \quad B \epsilon \mathcal{B},$$

then ξ_n _converges in distribution to some random variable_ ξ _satisfying_

(5.4) $P\{x \leq \xi\}=\wedge\{c(B);\ B \epsilon \mathcal{B},\ \uparrow x \subseteq F \subseteq B$ for some $F \epsilon \mathcal{L}\}$, $x \epsilon L$;

(5.5) $P\{\xi \epsilon F\}=\vee\{c(B);\ B \epsilon \mathcal{B},\ B \subseteq \uparrow x \subseteq F$ for some $x \epsilon L\}$, $F \epsilon \mathcal{L}$.

Thus, $\xi_n \xrightarrow{d} \xi$ if $P\{x \leq \xi_n\} \to P\{x \leq \xi\}$ for all x in a separating subset of L, or if $P\{\xi_n \epsilon F\} \to P\{\xi \epsilon F\}$ for all F in a separating subset of $\mathcal{L}$.

Proof: Suppose $\xi_n \xrightarrow{d} \xi$ and let $y \ll x$. Then $x \epsilon F \subseteq \uparrow y$ for some $F \epsilon \mathcal{L}$. By Lemma 5.3, there is a pair $(z,H) \epsilon L x \mathcal{L}$ satisfying $x \epsilon H \subseteq \uparrow z \subseteq F$ and $P\{\xi \epsilon \uparrow z \backslash H\}=0$. By Theorem 5.2, $P\{z \leq \xi_n\} \to P\{z \leq \xi\}$. Hence, the set of all $x \epsilon L$ for which

$$P\{x \leq \xi\}=\lim_n P\{x \leq \xi_n\}$$

is separating. Similarly, the reader may prove that the set of all $F \epsilon \mathcal{L}$ satisfying

$$P\{\xi \epsilon F\}=\lim_n P\{\xi_n \epsilon F\}$$

is separating.

To see the next part of the proposition, fix $\epsilon>0$ and choose $B \epsilon \mathcal{B}$, $y \epsilon L$ such that $B \subseteq \uparrow y$ and $c(B)>1-\epsilon$. Then $P\{\xi_n \epsilon B\}>1-\epsilon$ and, therefore, $P\{y \leq \xi_n\}$ for $n \geq n_0$, say. It follows easily that $\{\xi_n\}$ is tight. Hence there is a subsequence $\{\xi_{n(k)}\}$ and a random variable ξ such that $\xi_{n(k)} \xrightarrow{d} \xi$.

Fix $x \epsilon L$. If $y \ll x$ then $\uparrow x \subseteq B \subseteq \uparrow y$ for some $B \epsilon \mathcal{B}$. Hence, by (5.3) and Theorem 5.2,

$$\limsup_n P\{x \le \xi_n\} \le c(B) \le \limsup_k P\{y \le \xi_{n(k)}\} \le P\{y \le \xi\}.$$

By Proposition 4.1,

$$\limsup_n P\{x \le \xi\} \le P\{x \le \xi\}, \quad x \in L.$$

Similarly the reader may show

$$\liminf_n P\{\xi_n \in F\} \ge P\{\xi \in F\}, \quad F \in \mathcal{L}.$$

We conclude by Theorem 5.2 that $\xi_n \xrightarrow{d} \xi$.

To see (5.4), fix $x \in L$ and let $\epsilon > 0$ be arbitrary. By Proposition 4.1, there exists some $F_1 \in \mathcal{L}$ with $x \in F_1$ and

$$P\{x \le \xi\} \le P\{\xi \in F_1\} \le P\{x \le \xi\} + \epsilon.$$

But then $x \in F_2 \subseteq B \subseteq F_1$ for some $F_2 \in \mathcal{L}$ and $B \in \mathcal{B}$. Of course we may assume here that

$$P\{\xi \in F_i\} = \lim_n P\{\xi_n \in F_i\}, \quad i=1,2.$$

Now we get

$$P\{x \le \xi\} \le P\{\xi \in F_2\} = \lim_n P\{\xi_n \in F_1\} \le c(B) \le \lim_n P\{\xi_n \in F_2\}.$$

This shows (5.4). The proof of (5.5) is similar and left to the reader. QED

6. Infinite divisibility

Here we investigate the property of having a distribution which is infinitely divisible w r t the meet for random variables in a continuous semi-lattice. The related question of convergence in distribution of finite meets of independent random variables forming a null-array is treated too. We begin by writing down some definitions.

Let ξ be a random variable in a continuous semi-lattice L. We assume that L has a top, denoted 1. Say that ξ is <u>infinitely divisible</u> if, for all $n \in N$, we have

$$\xi \overset{d}{=} \wedge_{i=1}^{n} \xi_i$$

for some independent and identically distributed random variables $\xi_1, \ldots, \xi_n$.

Let $\{\xi_{nj};\ n \in N,\ 1 \leq j \leq m_n\}$ be a triangular array of random variables in L. (The m_n's are assumed to be finite.) We say that the ξ_{nj}'s form a <u>null-array</u> if they are independent for each fixed n and if

(6.1) $$\lim_n \sup_j P\{\xi_{nj} \in F\} = 0, \quad F \in \mathcal{L}.$$

Note that $\mathcal{L}$ is a base of neighborhoods of 1, which will be regarded as the point of infinitey. Accordingly we say that a measure μ on L is <u>locally finite</u> if $\mu L \setminus F < \infty$ for all $F \in \mathcal{L}$.

Let μ be a locally finite measure on L. Choose $\{F_n\} \subseteq \mathcal{L}$ such that $F_{n+1} \ll F_n$ for all n and $F_n \downarrow \{1\}$. Then, clearly,

$$\mu B \setminus \{1\} = \lim_n \mu B \setminus F_n, \quad B \in \Sigma.$$

Moreover, the measures $B \to \mu_n B = \mu B \setminus F_n$ are finite and such that

$$\mu_n F=\mu L\backslash(F\cap F_n)-\mu L\backslash F, \quad F\in\mathcal{L}.$$

Hence the restriction of μ to $L\backslash\{1\}$ is uniquely determined by the values $\mu L\backslash F$, $F\in\mathcal{L}$.

Our first result in this section identifies the collection of all locally finite measures on L supported by $L\backslash\{1\}$.

PROPOSITION 6.1: *Let L be a continuous semi-lattice with a top and a second countable Scott topology. Suppose μ is a locally finite measure on L. Put*

(6.2) $$\psi(F)=\mu L\backslash F, \quad F\in\mathcal{L}.$$

Then

(i) $\Delta_{F_1}\ldots\Delta_{F_n}\psi(F)\leq 0, \quad n\in N, \quad F,F_1,\ldots,F_n\in\mathcal{L};$

(ii) $\psi(F)=\lim_n\psi(F_n), \quad F,F_1,F_2,\ldots\in\mathcal{L}, \quad F_n\uparrow F;$

(iii) $\psi(L)=0.$

Conversely, let $\psi:\mathcal{L}\to R_+$. Suppose ψ satisfies conditions (i)-(iii) above. Then there exists a locally finite measure μ on L satisfying (6.2).

Proof: Let μ be a locally finite measure on L and define ψ by (6.2). Conditions (ii) and (iii) are obvious, so we only need to prove (i). However, (i) follows from

$$\Delta_{F_1}\ldots\Delta_{F_n}\psi(F)=-\mu F\cap F_1^c\cap\ldots\cap F_n^c,$$

the proof of which is straightforward, hence left to the reader.

Conversely, suppose $\psi:\mathcal{L}\to R_+$ satisfies conditions (i)-

(ii). Choose $\{F_n\}\subseteq L$ such that $F_{n+1}\ll F_n$ for every n and $F_n\downarrow\{1\}$. For $n\in N$ put

$$c_n(F)=(-\Delta_{F_n}\psi(F))/\psi(F_n), \quad F\in L.$$

Now use Theorem 4.3 to conclude that c_n extends to a probability measure λ_n on L. Put $\mu_n=\psi(F_n)\lambda_n$. Then μ_n is a finite measure on L satisfying

$$\mu_n L\setminus F=\psi(F)+\psi(F_n)-\psi(F\cap F_n), \quad F\in L.$$

Let us now put

$$\mu B=\lim_n \mu_n B, \quad B\in\Sigma.$$

Then μ is a locally finite measure on L satisfying (6.2).

QED

The main result in this section provides a Lévy-Khinchin representation of the infinitely divisible distributions on L.

THEOREM 6.2: Let L be a continuous semi-lattice with a top and a second countable Scott topology. The formulae

(6.3) $\qquad M=\cap\{F^c;\ F\in L,\ \xi\in F^c \text{ a s}\};$

(6.4) $\qquad \mu M\setminus H=-\log P\{\xi\in H\}, \quad H\in \mathrm{OFilt}(M)$

define a bijection between the set of all infinitely divisible distributions $P\xi^{-1}$ and the set of all pairs (M,μ), where $M=\downarrow x$ for some $x\in L$ while μ is a locally finite measure on M with $\mu\{x\}=0$.

Before the proof we remark that M is Scott closed, hence a continuous semi-lattice. Its Lawson dual is

$$\mathrm{OFilt}(M)=\{F\cap M;\ F\in L,\ F\cap M\neq\emptyset\}.$$

Note further that $\xi\in M$ a s and that, whenever $F\in L$, $F\cap M\neq\emptyset$ iff

$P\{\xi\in F\}>0$. As is easily seen these facts hold for all random variables ξ in L. One assertion of the theorem is that if ξ is infinitely divisible, then there is a point $x\in L$ satisfying $P\{\xi\leq x\}=1$ and such that $P\{y\leq\xi\}>0$ whenever $y\in L$, $y\ll x$.

We also state the following results:

PROPOSITION 6.3: *Let ξ be a random variable in L, and put*

(6.5) $$\psi(F)=-\log P\{\xi\in F\},\quad F\in\mathcal{L};$$

(6.6) $$\mathcal{L}_\psi=\{F\in\mathcal{L};\quad \psi(F)<\infty\}.$$

Then ξ is infinitely divisible iff $\mathcal{L}_\psi$ is a semi-lattice and

(6.7) $$\Delta_{F_1}\cdots\Delta_{F_n}\psi(F)\leq 0,\quad n\in N,\quad F,F_1,\ldots,F_n\in\mathcal{L}_\psi.$$

PROPOSITION 6.4: *Let $\psi:\mathcal{L}\to[0,\infty]$ and define $\mathcal{L}_\psi$ as in* (6.6). *Then there exists some infinitely divisible random variable ξ in L satisfying* (6.5) *iff $\mathcal{L}_\psi$ is a semi-lattice and, moreover,* (6.7) *holds together with*

(6.8) $$\psi(F)=\lim_n\psi(F_n),\quad F,F_1,F_2,\ldots\in\mathcal{L},\quad F_n\uparrow F;$$

(6.9) $$\psi(L)=0.$$

Proof of Theorem 6.2 and Propositions 6.3 and 6.4: Let ξ be a random variable in L, and define ψ and $\mathcal{L}_\psi$ by (6.5) and (6.6), resp. Suppose ξ is infinitely divisible. Then there are random variables $\xi_1,\xi_2,\ldots$ satisfying

$$P\{\xi\in F\}=P\{\xi_n\in F\}^n,\quad F\in\mathcal{L},\quad n\in N.$$

Note that $\psi(F)<\infty$ iff $P\{\xi\in F\}>0$, and in this case

$$\psi(F)=\lim_n nP\{\xi_n\notin F\}.$$

Now it is easily seen that $\mathcal{L}_\psi$ is a semi-lattice and that (6.7)

is at hand. This shows the necessity parts of Propositions 6.3 and 6.4.

Let us next suppose that L_ψ is a semi-lattice and that (6.7) is at hand. Clearly L_ψ is isomorphic to OFilt(M). We conclude that the latter is a semi-lattice with a top. It follows by a result of Lawson [10], recalled in Section 2, that M must have a top. That is, $M=\downarrow x$ for some $x \in L$.

Introduce

$$\psi_0(H)=-\log P\{\xi \in H\}, \quad H \in \mathrm{OFilt}(M).$$

If $H=F \cap M$ for some $F \in \mathcal{L}$, then $\psi_0(H)=\psi(F)$. Now the reader easily shows that ψ_0 fulfills the three conditions of Proposition 6.1. We conclude that there is a locally finite measure μ on M supported by $M \setminus \{x\}$ satisfying (6.4).

Thus the mapping $P\xi^{-1} \to (M,\mu)$, described in Theorem 6.2 is into. Since it is clearly one-to-one, it remains to be shown that it is onto.

For this, fix $M=\downarrow x$, where $x \in L$ and let μ be a locally finite measure on M with $\mu\{x\}=0$. Choose $\{H_n\} \subseteq \mathrm{OFilt}(M)$ such that $H_{n+1} \ll H_n$ for each n and $H_n \downarrow \{x\}$. For $n \in N$ write

$$\mu_n B=\mu B \setminus H_n, \quad B \in \Sigma_M.$$

Then conclude, as in the proof of Theorem 3-1-1 in Matheron [11], that the mapping $c_n:\mathrm{OFilt}(M) \to [0,1]$, given by

$$c_n(H)=\exp(-\mu_n M \setminus H), \quad H \in \mathrm{OFilt}(M),$$

satisfies the three conditions of Theorem 4.3. Let us put

$$c(H)=\exp(-\mu M \setminus H), \quad H \in \mathrm{OFilt}(M).$$

Then, as $n \to \infty$, $c_n(H) \to c(H)$ for all $H \in \mathrm{OFilt}(M)$. Of course c

satisfies the conditions of Theorem 4.3. Hence there is a random variable ξ in M satisfying (6.4). Note that $\Sigma_M = \Sigma_L \cap M \subseteq \Sigma_L$, since $M \subseteq \Sigma_L$. Hence we may regard ξ as a random variable in L concentrated on M. Of course ξ is infinitely divisible. Thus the mapping $P\xi^{-1} \to (M,\mu)$ is onto. Theorem 6.2 is proved. Clearly so is also the remaining part of Proposition 6.3.

To see the sufficiency part of Proposition 6.4, form

$$M = \cap\{F^c;\ F \in \mathcal{L},\ \psi(F) = \infty\}.$$

Being a Scott closed subset of L, M is a continuous semi-lattice. It is a routine exercise to show that

$$\mathrm{OFilt}(M) = \{F \cap M;\ F \in \mathcal{L},\ F \cap M \neq \emptyset\}.$$

Moreover, $F \cap M \neq \emptyset$ iff $\psi(F) < \infty$. Hence OFilt(M) and $\mathcal{L}_\psi$ are isomorphic. Thus also the former is a semi-lattice. Since it trivially has a top, we conclude that M has a top too [10]. We may now conclude by Proposition 6.1 that there exists a locally finite measure μ on M satisfying $\mu M \setminus F = \psi(F)$, $F \in \mathcal{L}$, $F \cap M \neq 0$. By the already proved Theorem 6.2, there exists an infinitely divisible random variable ξ satisfying (6.5). QED

Now assume that L is a continuous lattice. It is not hard to see that a measure μ on L is locally finite iff $\mu L \setminus \uparrow x < \infty$ for all $x \in L$ with $x \ll 1$. In this case the set

$$\{x \in L;\ \mu L \setminus \uparrow x < \infty\}$$

is closed under finite non-empty joins. Its join is 1.

<u>THEOREM 6.5</u>: <u>Let</u> L <u>be a continuous lattice</u>. <u>Suppose</u> Scott(L) <u>has a countable open base</u>. <u>The formulae</u>

(6.10) $$x = \vee\{y \in L;\ P\{y \leq \xi\} > 0\};$$

(6.11) $\qquad \nu{\downarrow}x\setminus{\uparrow}y=-\log P\{y\le\xi\}, \quad y\in L, \quad y\ll x$

define a bijection between the set of all infinitely divisible distributions $P\xi^{-1}$ *and the set of all pairs* (x,ν), *where* $x\in L$ *and* ν *is a locally finite measure on* ${\downarrow}x$ *with* $\nu\{x\}=0$.

PROPOSITION 6.6: *Let* ξ *be a random variable in* L *and define*

(6.12) $\qquad \phi(x)=-\log P\{x\le\xi\}, \quad x\in L;$

(6.13) $\qquad L_\phi=\{x\in L;\ \phi(x)<\infty\}.$

Then ξ *is infinitely divisible iff* L_ϕ *is closed under non-empty finite joins and*

(6.14) $\qquad \phi_n(x;x_1,\ldots,x_n)\le 0, \quad n\in N, \quad x,x_1,\ldots,x_n\in L$.

PROPOSITION 6.7: *Let* ξ *be an infinitely divisible random variable in* L *satisfying* $P\{x\le\xi\}>0$ *for all* $x\in L$ *with* $x\ll 1$. *Let* μ *be the locally finite measure on* L satisfying

$$\mu L\setminus{\uparrow}x=-\log P\{x\le\xi\}, \quad x\in L.$$

Then $\xi\overset{d}{=}\wedge_n\xi_n$, *where* $\xi_1,\xi_2,\ldots$ *are the atoms of a Poisson process on* L *with intensity* μ.

Proof of Theorem 6.5: Let ξ be an infinitely divisible random variable in L and define x by (6.10). Define further M and μ by (6.3) and (6.4), resp. Fix $y\in L$. Suppose $P\{y\le\xi\}>0$. If $y\in F\in \mathcal{L}$ then $P\{\xi\in F\}\ge P\{y\le\xi\}>0$. Hence $F\cap M\ne\emptyset$. But ${\uparrow}y=\cap_{y\in F\in\mathcal{L}}F$ and, therefore, $y\in M$. We conclude that $x\in M$. Next, suppose $y\ll\vee M$. Then $\vee M\in F\subseteq{\uparrow}y$ for some $F\in\mathcal{L}$. Clearly $P\{y\le\xi\}\ge P\{\xi\in F\}>0$. The latter inequality since $F\cap M\ne\emptyset$. This shows that

$$\{y;\ y \ll \vee M\} \subseteq \{y;\ P\{y \le \xi\} > 0\}.$$

Hence $\vee M \le x$, i e $M \subseteq \downarrow x$. Thus we have $\downarrow x = M$.

Now fix $y \in L$, $y \ll x$. Choose $\{H_n\} \in \mathrm{OFilt}(M)$ such that $H_n \downarrow (\uparrow y)$. Then

$$\mu M \setminus \uparrow y = \lim_n \mu M \setminus H_n = \lim_n -\log P\{\xi \in H_n\} = -\log P\{y \le \xi\}.$$

We may now conclude that (6.11) defines a measure which coincides with μ. QED

<u>Proof of Proposition 6.6</u>: The proof of the necessity part is analogous to the proof of the corresponding part of Proposition 6.3. It can safely be left to the reader. To see the sufficience, suppose that L_ϕ is closed under finite joins and that (6.14) is at hand. Define ψ and L_ψ by (6.5) and (6.6), resp.

Suppose $F_1, F_2 \in L$. By Proposition 4.1 we may choose $y_i \in F_i$ with $P\{y_i \le \xi\} > 0$, $i=1,2$. But $y_1 \vee y_2 \in F_1 \cap F_2$. Hence

$$P\{\xi \in F_1 \cap F_2\} \ge P\{y_1 \vee y_2 \le \xi\} > 0.$$

Thus L_ψ is a semi-lattice.

To see that (6.14) implies (6.7), argue as in the proof of Lemma 4.19. QED

<u>Proof of Proposition 6.7</u>: It is enough to note here that $x \le \wedge_n \xi_n$ iff there are no points of the Poisson process in the set $L \setminus \uparrow x$. The probability of the latter event is $\exp(-\mu L \setminus \uparrow x)$. QED

Let us now turn our attention to the convergence in distribution of finite meets of independent random variables forming a null-array.

<u>THEOREM 6.8:</u> <u>Let</u> $\{\xi_{nj}\}$ <u>be a null-array of random variables in</u>

a continuous semi-lattice L. *Suppose* L *is closed under finite non-empty joins and that* Scott(L) *is second countable. Let* ξ *be a random variable in* L. *Then* $\wedge_j \xi_{nj} \xrightarrow{d} \xi$ *iff*

$$(6.15)\qquad \begin{cases} \limsup_n \Sigma_j P\{\xi_{nj} \notin F\} \le -\log P\{\xi \in F\}, & F \in \mathcal{L} \\ \liminf_n \Sigma_j P\{x \not\le \xi_{nj}\} \ge -\log P\{x \le \xi\}, & x \in L. \end{cases}$$

Moreover, if $\wedge_j \xi_{nj} \xrightarrow{d} \xi$ *then* ξ *is infinitely divisible and there are separating subsets* A *and* $\mathcal{A}$ *of* L *and* $\mathcal{L}$, *resp. with*

$$(6.16)\qquad \lim_n \Sigma_j P\{\xi_{nj} \notin F\} = -\log P\{\xi \in F\}, \quad F \in \mathcal{A};$$

$$(6.17)\qquad \lim_n \Sigma_j P\{x \not\le \xi_{nj}\} = -\log P\{x \le \xi\}, \quad x \in A.$$

Conversely, $\wedge_j \xi_{nj} \xrightarrow{d} \xi$ *if* $\Sigma_j P\{\xi_{nj} \notin F\} \to -\log P\{\xi \in F\}$ *for all* F *in some separating subset of* $\mathcal{L}$, *or if* $\Sigma_j P\{x \not\le \xi_{nj}\} \to -\log P\{x \le \xi\}$ *for all* x *in some separating subset of* L.

Proof: It is a routine exercise to show that $P\{\wedge_j \xi_{nj} \in F\} \to P\{\xi \in F\}$ iff $\Sigma_j P\{\xi_{nj} \notin F\} \to -\log P\{\xi \in F\}$ and that $P\{x \le \wedge_j \xi_{nj}\} \to P\{x \le \xi\}$ iff $\Sigma_j P\{x \not\le \xi_{nj}\} \to -\log P\{x \le \xi\}$. (The latter provided $x \ll 1$ of course. Note that if $A \subseteq L$ is separating, then so is $A' = \{x \in A;\ x \ll 1\}$.) By Proposition 5.4, this holds for all x and F in separating subsets A and $\mathcal{A}$ of L and $\mathcal{L}$, resp, if $\wedge_j \xi_{nj} \xrightarrow{d} \xi$. Suppose this. Let $F \in \mathcal{L}$. Choose $\{F_m\} \subseteq \mathcal{A}$ such that $F_m \uparrow F$. Then

$$\Sigma_j P\{\xi_{nj} \notin F\} \le \Sigma_j P\{\xi_{nj} \notin F_m\} \to -\log P\{\xi \in F_m\}.$$

Hence

$$\limsup_n \Sigma_j P\{\xi_{nj} \notin F\} \le -\log P\{\xi \in F_m\} \to -\log P\{\xi \in F\}.$$

Similarly the reader may show that

$$\liminf_n \Sigma_j P\{x \not\le \xi_{nj}\} \ge -\log P\{x \le \xi\}.$$

Thus, (6.15) follows from (6.16) and (6.17), which follow from $\wedge_j \xi_{nj} \xrightarrow{d} \xi$. Conversely, the reader easily shows that

(6.15) implies both (6.16) and (6.17). By Proposition 5.4, each of these conditions imply $\wedge_j \xi_{nj} \xrightarrow{d} \xi$. QED

7. Applications to random set theory

Let S be a quasi locally compact second countable space and write $\mathcal{G}$ for its topology which is a continuous lattice under inclusion $\subseteq$. It is not hard to see that any open base for $\mathcal{G}$ is a separating subset provided it is closed under finite unions. Hence $\mathcal{G}$ contains a countable separating subset. By Proposition 3.1, Scott($\mathcal{G}$) is second countable.

By a <u>random open set</u> in S we understand a measurable $\mathcal{G}$-valued mapping of some probability space $(\Omega,\mathcal{R},P)$. Thus, by Proposition 3.2, $\xi:\Omega\to\mathcal{G}$ is a random open set iff $\{G\subseteq\xi\}\in\mathcal{R}$, $G\in\mathcal{G}$ iff $\{G\ll\xi\}\in\mathcal{R}$, $G\in\mathcal{G}$. By Theorem 4.2, the distribution of a random open set ξ is completely determined by its distribution function $\Lambda(G)=P\{G\subseteq\xi\}$, $G\in\mathcal{G}$. Now let $\Lambda:\mathcal{G}\to[0,1]$ be arbitrary. Then, by Theorem 4.4, there is a random open set in S with distribution function Λ iff (i) $\Lambda_n(G;G_1,\ldots,G_n)\geq 0$ for $n\in N$ and $G,G_1,\ldots,G_n\in\mathcal{G}$, (ii) $\Lambda(G_n)\to\Lambda(G)$ as $G_n\uparrow G$ and (iii) $\Lambda(\emptyset)=1$.

Let $\xi,\xi_1,\xi_2,\ldots$ be random open sets in S. Then $\xi_n\overset{d}{\to}\xi$ iff

$$\begin{cases}\liminf_n P\{G\ll\xi_n\}\geq P\{G\ll\xi\}, & G\in\mathcal{G};\\ \limsup_n P\{G\subseteq\xi_n\}\leq P\{G\subseteq\xi\}, & G\in\mathcal{G}.\end{cases}$$

Cf Theorem 5.2. Moreover, by Proposition 5.4, $\xi_n\overset{d}{\to}\xi$ if $P\{G\subseteq\xi_n\}\to P\{G\subseteq\xi\}$ for all G in a separating subset of $\mathcal{G}$.

The application of the results of Section 6 to this model of random open sets we leave to the reader.

Let us instead suppose that S is sober. Then S is locally compact [7]. Write $\mathcal{F}$ for the collection of all closed

sets in S. Endow $\mathcal{F}$ with the exclusion order $\supseteq$. Clearly $G \mapsto G^c$ is an isomorphism between $\mathcal{G}$ and $\mathcal{F}$. Hence (under exclusion) $\mathcal{F}$ is a continuous lattice with a second countable Scott topology. (Note that, for $\{F_\alpha\} \subseteq \mathcal{F}$, $\vee_\alpha F_\alpha = \cap_\alpha F_\alpha$.) Write further $\mathcal{K}$ for the collection of all compact and saturated sets in its natural order $\subseteq$. Note that, being isomorphic to the Lawson dual of $\mathcal{F}$, $\mathcal{K}^*$ is a continuous semi-lattice with top and second countable Scott topology (cf (2.6)).

Say that an $\mathcal{F}$-valued mapping of a probability space is a <u>random</u> <u>closed</u> <u>set</u> in S if it is measurable. By Proposition 3.2, $\xi : \Omega \to \mathcal{F}$ is a random closed set iff $\{\xi \subseteq F\} \in \mathcal{R}$, $F \in \mathcal{F}$ iff $\{\xi \cap K = \emptyset\} \in \mathcal{R}$, $K \in \mathcal{K}$. Clearly these conditions hold iff $\{\xi \cap G = \emptyset\} \in \mathcal{R}$, $G \in \mathcal{G}$. We see that in the particular case when S is a Hausdorff space our notion of a random closed set coincides with Matheron's [11]. Most of the subsequent results for random closed sets are well-known in this particular case. Cf also [2] [13].

Let us first note that, by Theorem 4.2, the distribution of a random closed set ξ is completely determined by the values $P\{\xi \cap G = \emptyset\}$, $G \in \mathcal{G}$ or $P\{\xi \cap K \neq \emptyset\}$, $K \in \mathcal{K}$. (The tradition invites us to work with the function $G \mapsto P\{\xi \cap G = \emptyset\}$, $G \in \mathcal{G}$ rather than with the distribution function $\Lambda(F) = P\{\xi \subseteq F\}$, $F \in \mathcal{F}$.)

Let $T : \mathcal{K} \to R$. Then, by Theorem 4.3, there is a random closed set ξ satisfying $P\{\xi \cap K \neq \emptyset\} = T(K)$, $K \in \mathcal{K}$ iff

(i) $T_n(K; K_1, \ldots, K_n) \leq 0$, $n \in N$, $K, K_1, \ldots, K_n \in \mathcal{K}$;

(ii) $T(K) = \lim_n T(K_n)$, $K, K_1, K_2, \ldots \in \mathcal{K}$, $K_n \downarrow K$;

(iii) $0 \leq T \leq 1$, $T(\emptyset) = 0$.

In the terminology of [11], (i)-(iii) hold iff "T is an alternating Choquet capacity of infinite order such that $0 \le T \le 1$ and $T(\emptyset)=0$". Thus, as claimed in the introduction, our Theorem 4.3 extends Choquet's existence theorem for distributions of random closed sets.

Next let $Q: \mathcal{G} \to [0,1]$. By Theorem 4.4 there is a random closed set ξ satisfying $P\{\xi \cap G=\emptyset\}=Q(G)$, $G \in \mathcal{G}$ iff

(i) $Q_n(G;G_1,\ldots,G_n) \ge 0$, $n \in N$, $G,G_1,\ldots,G_n \in \mathcal{G}$;

(ii) $Q(G)=\lim_n Q(G_n)$, $G,G_1,G_2,\ldots \in \mathcal{G}$, $G_n \uparrow G$;

(iii) $Q(\emptyset)=1$.

Also this existence result can be found in [11].

Before turning to the convergence in distribution of random closed sets, let us note that the Lawson topology on $\mathcal{F}$ is generated by the families $\{F \in \mathcal{F};\ F \cap K=\emptyset\}$, $K \in \mathcal{K}$ and $\{F \in \mathcal{F}; F \cap G \ne \emptyset\}$, $G \in \mathcal{G}$. Hence it coincides with Fell's topology [4]. See also [11] [13].

Let $\xi,\xi_1,\xi_2,\ldots$ be random sets in S. Then, by Theorem 5.2, $\xi_n \xrightarrow{d} \xi$ iff

$$\begin{cases} \liminf_n P\{\xi_n \cap K=\emptyset\} \ge P\{\xi \cap K=\emptyset\}, & K \in \mathcal{K} \\ \limsup_n P\{\xi_n \cap G=\emptyset\} \le P\{\xi \cap G=\emptyset\}, & G \in \mathcal{G}. \end{cases}$$

By Proposition 5.4, $\xi_n \xrightarrow{d} \xi$ if $P\{\xi_n \cap K \ne \emptyset\} \to P\{\xi \cap K \ne \emptyset\}$ for all K in a separating subset of $\mathcal{K}^*$ or if $P\{\xi_n \cap G=\emptyset\} \to P\{\xi \cap G=\emptyset\}$ for all G in a separating subset of $\mathcal{G}$. Note that $\mathcal{K}_0 \subseteq \mathcal{K}^*$ is separating iff, whenever $K \subseteq G$, where $G \in \mathcal{G}$ while $K \in \mathcal{K}$, we have $K \subseteq K_0 \subseteq G$ for some $K_0 \in \mathcal{K}_0$. It is not hard to verify that the collection $\mathcal{K}_\xi$ of all $K \in \mathcal{K}$ with $P\{\xi \cap K^o=\emptyset\}=P\{\xi \cap K=\emptyset\}$ is a separating subset of $\mathcal{K}^*$ (use Lemma 5.3). Hence $\xi_n \xrightarrow{d} \xi$ iff

$$P\{\xi\cap K\neq\emptyset\}=\lim_n P\{\xi_n\cap K\neq\emptyset\}, \quad K\in K_\xi.$$

Cf [13].

By applying the results displayed above for random closed sets to the case when S equals the extended real line $(-\infty,\infty]$ endowed with the topology with non-trivial closed sets $(-\infty,x]$, $x\in R$, we may obtain some very familiar existence and convergence results for distributions of random variables. This is left to the reader. Note that this topology on $(-\infty,\infty]$ is not Hausdorff. Hence Choquet's original result can not be applied. However it is locally compact, second countable and sober. This can be seen either directly or by noting that it coincides with the Scott topology on $(-\infty,\infty]$. Note that these results for random variables also can be obtained directly from the results of the Sections 4 and 5.

The application of the results of Section 6 to random closed sets is left to the reader (cf with [11] [13]).

Let us agree to say that a K-valued mapping of some probability space is a <u>random compact set</u> if it is measurable as a mapping into K^*. That is $\xi:\Omega\to K$ is a random compact set iff $\{\xi\subseteq K\}\in R$, $K\in K$. By Proposition 3.2, this holds iff $\{\xi\subseteq K^o\}\in R$, $K\in K$. By Theorem 4.2, the distribution of a random compact set ξ is completely determined by its distribution function $\Lambda(K)=P\{\xi\subseteq K\}$, $K\in K$ or the values $P\{\xi\subseteq K^o\}$, $K\in K$.

Let $c:G\to[0,1]$. By Theorem 4.3 there is a random compact set ξ satisfying $P\{\xi\subseteq G\}=c(G)$, $G\in G$ iff

(i) $\Delta_{G_1}\cdots\Delta_{G_n} c(G)\geq 0$, $n\in N$, $G,G_1,\ldots,G_n\in G$;

(ii) $c(G)=\lim_n c(G_n)$, $G,G_1,G_2,\ldots \in \mathcal{G}$, $G_n \uparrow G$;

(iii) $c(S)=1$.

Note the similarity with the second of the existence theorems displayed above for random closed sets.

Now we tighten the assumptions on S somewhat further and suppose that S is a Hausdorff space. Then all subsets of S are saturated and, in particular, $\mathcal{K}$ consists of all compact sets in S. Note that $\mathcal{K}$ now is closed under all non-empty intersections and finite unions.

Let $\Lambda:\mathcal{K}\to[0,1]$. Then, by the remark immediately following Theorem 4.4, there exists a random compact set in S satisfying $P\{\xi\subseteq K\}=\Lambda(K)$, $K\in\mathcal{K}$ iff

(i) $\Delta_{K_1}\cdots\Delta_{K_n}\Lambda(K)\geq 0$, $n\in N$, $K,K_1,\ldots,K_n\in\mathcal{K}$;

(ii) $\Lambda(K)=\lim_n\Lambda(K_n)$, $K,K_1,K_2,\ldots\in\mathcal{K}$, $K_n\downarrow K$;

(iii) $\vee_{K\in\mathcal{K}}\Lambda(K)=1$.

Note that if S is compact then (iii) reduces to $\Lambda(S)=1$. This second set of existence criteria for random compact sets should be compared with the first of the existence results for random closed sets given above.

Let $\xi,\xi_1,\xi_2,\ldots$ be random compact sets in S. By Theorem 5.2, $\xi_n\xrightarrow{d}\xi$ iff

$$\begin{cases}\liminf_n P\{\xi_n\subseteq K^o\}\geq P\{\xi\subseteq K^o\}, & K\in\mathcal{K}\\ \limsup_n P\{\xi_n\subseteq K\}\leq P\{\xi\subseteq K\}, & K\in\mathcal{K}.\end{cases}$$

By Proposition 5.4, $\xi_n\xrightarrow{d}\xi$ if $P\{\xi_n\subseteq G\}\to P\{\xi\subseteq G\}$ for all G in a separating subset of $\mathcal{G}$ or if $P\{\xi_n\subseteq K\}\to P\{\xi\subseteq K\}$ for all K in a separating subset of $\mathcal{K}^*$. It follows easily by Lemma 5.3 that the collection of all $K\in\mathcal{K}$ with $P\{\xi\subseteq K\}=P\{\xi\subseteq K^o\}$ is a separating

subset of $\mathcal{K}^*$. Hence $\xi_n \xrightarrow{d} \xi$ iff $P\{\xi_n \subseteq K\} \to P\{\xi \subseteq K\}$ for all K in this separating subset.

Let us recall here that the convergence in distribution is w r t the Lawson topology and that this topology is generated by the families $\{K \in \mathcal{K};\ K \subseteq M\}$, $M \in \mathcal{K}$ and $\{K \in \mathcal{K};\ K \subseteq M^\circ\}$, $M \in \mathcal{K}$.

Let ξ be a random compact set in S. Then ξ is infinitely divisible iff, whenever $n \in N$, we have $\xi \stackrel{d}{=} \bigcup_{i=1}^{n} \xi_i$ for some independent and identically distributed $\xi_1, \ldots, \xi_n$. Put

(7.1) $\qquad \psi(G) = -\log P\{\xi \subseteq G\}, \quad G \in \mathcal{G};$

(7.2) $\qquad \mathcal{G}_\psi = \{G \in \mathcal{G};\ \psi(G) < \infty\}.$

By Proposition 6.3, ξ is infinitely divisible iff $\mathcal{G}_\psi$ is a semi-lattice and

(7.3) $\qquad \Delta_{G_1} \cdots \Delta_{G_n} \psi(G) \leq 0, \quad n \in N, \quad G, G_1, \ldots, G_n \in \mathcal{G}_\psi.$

Now let $\psi: \mathcal{G} \to [0,\infty]$ be arbitrary and define $\mathcal{G}_\psi$ by (7.2). Then there exists an infinitely divisible random compact set ξ satisfying (7.1) iff $\mathcal{G}_\psi$ is a semi-lattice and, moreover, (7.3) holds together with

(7.4) $\qquad \psi(G) = \lim_n \psi(G_n), \quad G, G_1, G_2, \ldots \in \mathcal{G}, \quad G_n \uparrow G;$

(7.5) $\qquad \psi(S) = 0.$

Cf Proposition 6.4.

Finally we assume that $S = R^d$ for some $d \in N$. Let $\mathcal{C}$ be the collection of all compact and convex subsets of S. We regard $\emptyset$ as convex. Note that non-empty intersections of convex sets are convex. It follows, for $C_1, C_2 \in \mathcal{C}$, $C_1 \ll C_2$ w r t the exclusion order iff $C_2 \subseteq C_1^\circ$. Now it is easily seen that $\mathcal{C}^*$ is a continuous poset which is closed under finite non-empty

joins. Moreover, $\mathcal{C}^*$ is closed under finite non-empty meets too. Since the meet of $\{C_1,C_2\}$ w r t exclusion coincides with the convex hull of $C_1 \cup C_2$.

Say that a $\mathcal{C}$-valued mapping of some probability space is a random convex set if it is measurable w r t $\mathcal{C}^*$. By Proposition 3.2, $\xi:\Omega\to\mathcal{C}$ is a random convex set in S iff $\{\xi\subseteq C\}\in\mathcal{R}$, $C\in\mathcal{C}$ iff $\{\xi\subseteq C^o\}\in\mathcal{R}$, $C\in\mathcal{C}$. By Proposition 4.2, the distribution of a random convex set is uniquely determined by its distribution function $\Lambda(C)=P\{\xi\subseteq C\}$, $C\in\mathcal{C}$ or by the values $P\{\xi\subseteq C^o\}$, $C\in\mathcal{C}$. Now let $\Lambda:\mathcal{C}\to[0,1]$ be arbitrary. Then there is a random convex set ξ satisfying $P\{\xi\subseteq C\}=\Lambda(C)$, $C\in\mathcal{C}$ iff

(i) $\Delta_{C_1}\cdots\Delta_{C_n}\Lambda(C)\geq 0$, $n\in N$, $C,C_1,\ldots,C_n\in\mathcal{C}$;

(ii) $\Lambda(C)=\lim_n\Lambda(C_n)$, $C,C_1,C_2,\ldots\in\mathcal{C}$, $C_n\downarrow C$;

(iii) $\vee_{C\in\mathcal{C}}\Lambda(C)=1$.

See the remark after Theorem 4.4.

The Lawson topology on $\mathcal{C}$ is generated by the two families $\{C\in\mathcal{C};\ C\subseteq D\}$, $D\in\mathcal{C}$ and $\{C\in\mathcal{C};\ C\subseteq D^o\}$, $D\in\mathcal{C}$.

Let ξ,ξ_1,ξ_2 be random convex sets in S. Then, by Theorem 5.2, $\xi_n\overset{d}{\to}\xi$ iff

$$\begin{cases}\liminf_n P\{\xi_n\subseteq C^o\}\geq P\{\xi\subseteq C^o\}, & C\in\mathcal{C};\\ \limsup_n P\{\xi_n\subseteq C\}\leq P\{\xi\subseteq C\}, & C\in\mathcal{C}\end{cases}$$

Moreover, by Proposition 5.4, $\xi_n\overset{d}{\to}\xi$ if $P\{\xi_n\subseteq C\}\to P\{\xi\subseteq C\}$ for all C in some separating subset of $\mathcal{C}$. Note that $\mathcal{B}\subseteq\mathcal{C}$ is separating iff whenever $C_1\subseteq C_2^o$ we have $C_1\subseteq C\subseteq C_2$ for some $C\in\mathcal{B}$. The collection of all $C\in\mathcal{C}$ for which $P\{\xi\subseteq C^o\}=P\{\xi\subseteq C\}$ is a separating class. See Lemma 5.3. We may now conclude that $\xi_n\overset{d}{\to}\xi$ iff $P\{\xi_n\subseteq C\}\to P\{\xi\subseteq C\}$ for

all $C \in \mathcal{C}$ with $P\{\xi \subseteq C^0\} = P\{\xi \subseteq C\}$.

Note that a random convex set ξ is infinitely divisible iff for each $n \in N$ there exists independent and identically distributed random convex sets $\xi_1, \ldots, \xi_n$ such that the distribution of the convex hull of $\xi_1 \cup \ldots \cup \xi_n$ coincides with the distribution of ξ. By Theorem 6.5, the formulae

$$B = \cap \{C \in \mathcal{C};\ P\{\xi \subseteq C\} > 0\};$$

$$\nu\{C \in \mathcal{C};\ B \subseteq C,\ C \subseteq D\} = -\log P\{\xi \subseteq D\},\ D \in \mathcal{C},\ B \subseteq D^0$$

define a unique correspondence between the set of all infinitely divisible distributions $P\xi^{-1}$ on $\mathcal{C}' = \mathcal{C} \cup \{S\}$, and the set of all pairs (B, ν), where $B \in \mathcal{C}'$ while ν is a locally finite measure on $\{C \in \mathcal{C};\ B \subseteq C\} \cup \{S\}$ with $\nu\{B\} = 0$. (Since $\mathcal{C} \cup \{S\}$ endowed with exclusion is a continuous lattice.) Nete in connection with this characterization of the infinitely divisible distributions on $\mathcal{C}'$ that a random variable ξ in $\mathcal{C}'$ is supported by $\mathcal{C}$ iff

$$\vee_{C \in \mathcal{C}} P\{\xi \subseteq C\} = 1.$$

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